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# ON A THEORY OF RATES


**Bruce W. Fowler**

**Advanced Science and Technology Directorate**
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While the concept of Rate is common to many disciplines, there exists no consistent and general mathematical theory of rates in any discipline. In this report, a general theory of rates is developed as a transformation of a discrete or punctuated representation into a continuous representation using functional theory. Stochastic events are represented using Renewal Theory and Order Statistics. Examples of several rate differential equations are presented, in some cases differing considerably from their equivalent ad hoc developments.

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**TABLE OF CONTENTS**

## LIST OF ILLUSTRATIONS

## LIST OF TABLES

## I. INTRODUCTION

Rate is a concept common to almost all scientific and technical disciplines. The **Oxford Desk Dictionary** [Abate 1997] defines *rate* as a "numerical proportion expressed in units" and as a "pace of movement or change." These definitions are not very helpful, but they do convey the deceptive simplicity of the idea.

Despite this almost pervasiveness, there is little in the way of a common theory of rates. In particular, the various formulations of rates are often imprecise and *ad hoc*. Further, the rate differential equation, an inseparable part of the basic concept, is almost always a *deus ex machina*. As a result, rate formulations are often arcane and less accurate than they could be.

Our purpose here is to investigate and elaborate the nucleus of just such a general theory of rates. This theory is new and incomplete, but it offers great potential for the understanding and application of rates.

This body of methodology was first introduced in [Fowler 2001] and then elaborated in [Fowler 2004] to expand the nature of attrition rate differential equations.

## II. WHAT'S A RATE?

Regardless of whether we are describing the collisions of photons with other particles, the chemical interaction of molecules, or the populations of sexually or asexually reproducing animals, we are dealing with events that are well localized in time; that is, they take less time to occur than the period of observation. To some degree, we may view these events as occurring impulsively or instantaneously, although we shall almost immediately generalize the theory to accommodate events that have a finite duration.

This leads us to a third definition of rate: *rate* is a means of continuously representing the change in quantity of a population of discrete events or other observables. By this we mean that a rate is a mapping of discrete events into a continuous representation.

Actually, we want to also consider the opposite as well, the mapping of a continuous process into discreteness. For simplicity at this point, we shall assume this complementary aspect is merely a matter of piecewise integration, however deceptively.

Simply put, however, a rate is a mathematical expression of how many things take place in a unit of time. As such, it is a change of representation from a collective discrete representation of when each of the things occur to a continuous representation. Ideally this transformation reproduces the occurrence times of the events accurately, although the very concept of a rate implies some sort of smoothing or averaging of event times.

We advance a set of propositions to lay the groundwork for a general theory of rates:

**Proposition 1.** A rate is a mapping of a collection or series of connected but discrete events or observations into a continuous counting representation.

This is essentially a restatement of the basic definition above in somewhat more mathematical terms. The key point is that we are starting with discrete, that is, punctuated in position (normally time), events that we transform into some continuous representation. The continuous representation is an enumeration of the events even though it is no longer integer.

**Proposition 2.** Observable events consist of at least two measurable quantities: the immediate intensity arising from the occurrence of the event (which may be just the occurrence) and the position of the event.

Each of these punctuated events has a magnitude and a position. That is, we know how big some aspect of the event is. Assuming the position is temporal, we know when the event occurs.

At this point, we have not yet addressed whether the duration of the event is finite (as opposed to infinitesimal.) We further assume that these two quantities are either completely independent of each other, or if they are dependent, we have a model of the nature of the dependency.

**Proposition 3.** Events occur as the effect of some cause.

This is not some supernatural phenomenon. We assume that the basic scientific relation of cause and effect appertain. We further assume that we know this relationship.

**Proposition 4.** Events may be caused by single or multiple agents.

We assume that the causality can be systematized; that it is not holistic. Specifically, we assume that the cause agents may be aggregable.

**Proposition 5.** Events may be described by some mathematical formalism.

The mechanics of the events has some mathematical description. This follows largely from the proposition of cause and effect, but it also says that things are fairly well behaved.

**Proposition 6.** The mechanism that causes events may be singular or repetitive.

This simply says that cause agents may cause one event or they may cause several in succession.

**Proposition 7.** Cause agents are independent and may be identical, but their association in a rate process has a collective effect.

This lets us treat the agents as if they are identical but independent. It also acknowledges that they have a different effect when they band together. Recall Napoleon's dictum "There is a quality in quantity."

**Proposition 8.** Events occur in a discrete manner although not necessarily limited to an infinitesimal change in position.

Although the events are punctuated, they may not occur instantaneously.

Our intent is captured in Figure 1 where we depict two discrete event trajectories (they look rather like stairs,) and two equivalent continuous representations. In this case, these are taken from the application of General Rate Theory to attrition.

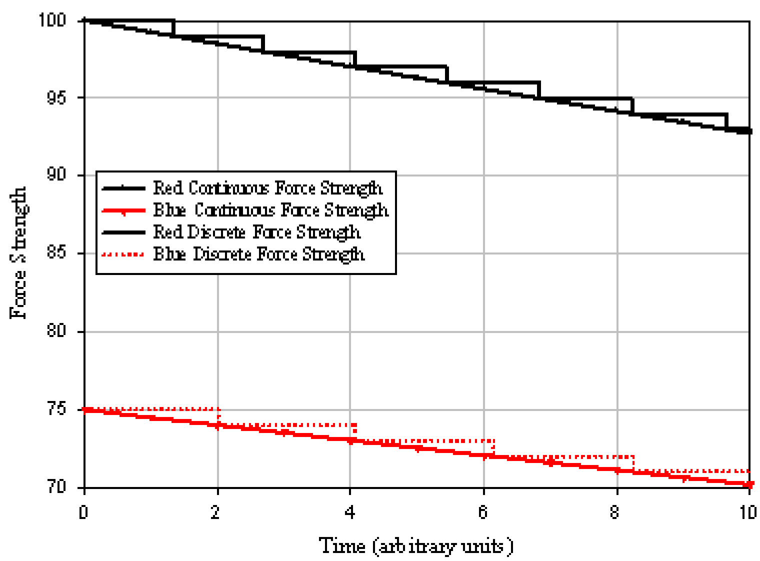


*Figure 1. Discrete Event and Continuous Representation Trajectories*

Our approach is to first briefly present the pieces of mathematics needed to develop rate theory. We then use these pieces to first develop the continuous representations of the discrete events and then the general rate differential equations. Finally, we present several examples of the application of the theory.

## III. FUNCTIONAL THEORY

Our starting point in developing this General Theory of Rates is what is often called Functional Theory [Stakgold 1998], although it is sometimes called the Theory of Generalized Functions [Erdelyi 1962] or Distribution Theory. One of the most common uses of this theory is the development of Green Functions. A Green Function is a representation of the behavior of a mechanical system under the imposition of an impulsive "force." Our approach is similar in that we are dealing with impulsive events. The difference is that the Green Function, via the vehicle of the properties of the Dirac delta function, is used to describe the behavior of a mechanical system under the imposition of a general "force," while our interest in a General Theory of Rates is finding one or more means of transforming a general collection of Dirac delta functions into a continuous representation.

The Dirac or strong delta function, $\delta(t-t')$, is not a function in the normal sense. In particular, it has the behavior

$$\delta(t-t') = \begin{cases} 0,\, t < t' \\ 1,\, t = t' \\ 0.\, t > t' \end{cases} \tag{1}$$

One then naturally asks the question, If the integral

$$\int_{t'-\varepsilon}^{t'+\varepsilon} \delta(t-t')\, dt' = 1, \varepsilon > 0, \tag{2}$$

then for what value of $\varepsilon' < \varepsilon$ does the equivalent integral

$$\int_{t'-\varepsilon'}^{t'+\varepsilon'} \delta(t-t')\, dt' = f \tag{3}$$

have a value of $0 < f < 1$? This is clearly conceivable, and our intuition tells us that such a situation must exist. The methodology or formalism for describing this is given by **Functional Theory**.

Delta functions have functional representations as Gaussians (among other forms) in a weak sense. By weak, we mean that if we consider

$$g(x : \overline{x}, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\overline{x})^2/2\sigma^2}, \tag{4}$$

which we know has unit integral, then

$$\lim_{\sigma \to 0} g(x : \overline{x}, \sigma) \to \delta(x - \overline{x}) \tag{5}$$

in the sense of Equation (1).

We may extend this idea even to the simple Negative Exponential Distribution (NED),

$$p\,(t:\alpha) = \alpha\, e^{-\alpha t}. \tag{6}$$

If we take the limit $\alpha \to \infty$, then

$$p\,(t:\alpha \to \infty) = \begin{cases} 1, t = 0, \\ 0,\ t > 0 \end{cases} \tag{7}$$

This is again a delta function. We shall refer to this behavior as a weak delta function (or functional) in contrast with a classical or strict sense.

The basic idea of functional (or distribution) theory is that instead of describing a function $f\,(x)$ with a value at every point $x$, we use the real number $\int\limits_{R_n} f\,(x)\ \phi\,(x)\ dx$ for every $\phi$ belonging to a class of accessory functions. In this representation, $f$ is a *functional* on the class of accessory functions. For developing rate theory, the class of accessory functions is comprised of the individual probability functions [Stakgold 1998].

The functional, designated by $\langle f, \phi \rangle$, is defined by the integral relationship

$$\langle f, \phi \rangle \equiv \int\limits_{R_n} f\,(x)\ \phi\,(x)\ dx, \tag{8}$$

which may be multi-dimensional. This integral spans the space of $f\,(x)$, and the accessory functions must span this space as well. The derivative of the functional is given by

$$\left\langle \frac{d}{dx} f, \phi \right\rangle = \left\langle f, -\frac{d}{dx}\phi \right\rangle. \tag{9}$$

In particular, we note that if the function $f\,(x)$ has discrete jumps that occur at points $\overline{x}_i$, then its functional derivative has the form

$$\frac{d}{dx} f = \left[\frac{d}{dx} f\right] + \sum_i \Delta f_i\, \delta\,(x - \overline{x}_i), \tag{10}$$

where $[f']$ is the piecewise continuous derivative and $\Delta f_i$ is the signed magnitude of the $i^{th}$ jump. We shall refer to this as the jump factor or function, depending on form.

Let us now consider the differential equation

$$\frac{du}{dx} = f. \tag{11}$$

If $f\,(x)$ is a continuous function, then we may find a classical (or strict) solution $u\,(x)$ that obeys the differential Equation (11) continuously at every point $x$. If, however, $f$ is a functional, then $u$ is a solution of the differential Equation (11) if and only if

$$-\left\langle u, \frac{d\phi}{dx} \right\rangle = \langle f, \phi \rangle \tag{12}$$

for all $\phi$ in the set that spans the space. In particular, if we want $u$ to be a function, then Equation (12) reduces to the differential Equation (11) in the weak sense. We must note, however, that any weak solution that has $m$ continuous derivatives, where $m$ is the dimension of the space, is a classical solution. For our purposes, we adopt this immediately as an assumption.

What this translates to is that if we can find a way to reduce the delta functions of Equation (10) to having a weak behavior, then we reduce that equation from a functional differential equation to a function (or classical) differential equation. In terms of our search for a General Theory of Rates, this differential equation is a (if not the) fundamental equation.

## IV. OVERVIEW OF LAPLACE TRANSFORMS

Our second body of theory, which is particularly useful in considering deterministic event times, is Laplace Transforms (LT) [Doetsch 1961].

Assume that $f(t)$ , a function, is well behaved and defined for $t \geq 0$. Then we may define the **Laplace Transform** of $f(t)$ as

$$\begin{aligned} L[f(t)] &= \int_0^\infty e^{-st} f(t)\, dt, \\ &\equiv f(s). \end{aligned} \tag{13}$$

The **Inverse Laplace Transform** is defined as

$$f(t) = \frac{1}{2\pi i} \lim_{\beta\to\infty} \int_{\gamma-i\beta}^{\gamma+i\beta} e^{st} f(s)\, ds, \tag{14}$$

which, unlike the Inverse Fourier Transform, is not symmetric and is considerably more difficult to perform than the transform itself. For this reason, a common practice with LT is to invert it by wimp inspection of tables of transforms rather than by macho integration.

The LT also obeys the convolution theorem,

$$L\left[\int_0^t f_1(t-t') f_2(t')\, dt'\right] = f_1(s) f_2(s). \tag{15}$$

Derivatives have the transform

$$L\left[\frac{d^n f(t)}{dt^n}\right] = s^n f(s) - \sum_{j=0}^{n-1} s^j \left.\frac{d^{n-j-1} f(t)}{dt^{n-j-1}}\right|_{t=0}, \tag{16}$$

and repeated integrals have the transform

$$L\left[\int_0^t dt_n .. \int_0^{t_2} dt_1 f(t_1)\right] = s^{-n} f(s). \tag{17}$$

Also useful are the moments,

$$L\left[t^n f(t)\right] = (-1)^n \frac{d^n f(s)}{ds^n}. \tag{18}$$

The expected value of a function is usually given as

$$\langle t \rangle = \int_0^\infty t' f(t')\, dt', \tag{19}$$

assuming $f(t)$ is defined on $[0, \infty)$. If we write the quantity $\int_0^\infty t' f(t')\, e^{-st'}\, dt'$, which is just the LT of $t\, f(t)$, then we see immediately that

$$\langle t \rangle = \lim_{s \to 0} L\left[t\, f(t)\right], \tag{20}$$

and by Equation (18), we may make the association

$$\langle t \rangle = -\lim_{s \to 0} \frac{df(s)}{ds}. \tag{21}$$

While multidimensional LT are possible, they seldom arise so we shall not treat with them here.

## V. PROBABILITY THEORY

If an event is stochastic, it has a **Cumulative Probability Function** or **Cumulative Distribution Function** (CDF) that we designate as a capital Roman letter function of time (for example, $P(t)$) and possibly other parameters or variables. This introduces an implicit assumption that the process producing the event is continuous and mathematically well behaved. In particular, we assume that the CDF is at least $m$ times differentiable. In some cases, notably with the combat processes in the next chapter, we shall deal with discrete probabilities, but they are still assumed to be well behaved.

Because of this differentiability, it is also useful to deal with the **Probability Density Function** (PDF) that we designate as a small Roman letter function of time (for example, $p(t)$) and possibly other parameters or variables. The CDF is the probability (we use likelihood as a synonym,) that an observable event has occurred by time $t$, while the PDF is the probability per unit time that the observable event has occurred within the infinitesimal interval $dt$ of time $t$. As indicated above, these are related as derivative and integral, so that we have the relationships,

$$\begin{aligned} p(t) &\equiv \frac{d}{dt} P(t), \\ P(t) &\equiv \int_0^t p(t')\, dt'. \end{aligned} \tag{22}$$

In addition to the Guassian and NED distributions already discussed, we also have the Gamma Distribution of (integer) order $n$ (also sometimes called the $n^{th}$ rank Erlang Distribution,) given by

$$p_n(t) \equiv \frac{\alpha}{(n-1)!} (\alpha t)^{n-1} e^{-\alpha t}. \tag{23}$$

This distribution has expected value and standard deviation of

$$\begin{aligned} \langle t \rangle &= \frac{n}{\alpha}, \\ \sigma &= \frac{\sqrt{n}}{\alpha}, \end{aligned} \tag{24}$$

which is sometimes referred to as a Poisson standard deviation since this distribution occurs in a Poisson process.

Yet another distribution we want to consider is the Lognormal Distribution given by

$$p(t) = \frac{1}{\sqrt{2\pi}\Sigma\, t} \exp\left[-\frac{(\ln(t) - \ln(\mu))^2}{2\,\Sigma^2}\right], \tag{25}$$

which is just a normal distribution of the logarithm of the random variable $t$. This distribution has expected value and standard deviation

$$\begin{aligned} \langle t \rangle &= \mu\, e^{-\Sigma^2/2}, \\ \sigma &= \mu \sqrt{e^{-\Sigma^2}\left(e^{-\Sigma^2} - 1\right)}, \end{aligned} \tag{26}$$

which is rather complicated and makes for some interesting conversion challenges. Notably the LT of the lognormal distribution does not have a simple representation. It is easy to calculate. All one has to do is expand the exponential $e^{-st}$ in an infinite series about zero in either variable. The integral is then calculated for each term and produces an exact result. Unfortunately, each term of $t^n$ produces a transform term of $e^{-n^2\Sigma^2/2}$. Thus the resulting series, which is the LT, is well defined but may not be compactly summed.

For deterministic events, we shall represent the PDF of the event by the Dirac (strong) delta function defined by Equation (1). In most cases, we shall be dealing with the translated delta function $\delta(t - t')$, which has non-zero value only at $t = t'$. The expected value and standard deviation of this translated delta function distribution are

$$\begin{aligned} \langle t \rangle &= t', \\ \sigma &= 0, \end{aligned} \tag{27}$$

which are what we should expect for a deterministic event. That is, the event has a definite time of occurrence.

Two elaborations of probability theory are needed to describe repeated events caused by the same actor and events caused by a group of actors. We now consider each in turn.

### A. Renewal Theory

Repetitive actions, that is repetitions of a process, result in a series of repetitive events. The mathematical formalism used to describe this repetition of a process is **Renewal Theory** [Ross 1970]. "A renewal (counting) process is a nonnegative integer-valued stochastic process that registers the successive occurrence of an event during a time interval, where the times between consecutive events are *positive, independent, identically distributed* random variables" [Taylor Karlin 1998].

For our purposes, we are primarily concerned with the repeated nature of the process and shall relax the identically distributed property while retaining the positive and independent properties. This generalization is consistent with both the mathematical theory of renewals and the problem at hand. In the absence, as yet, of detailed knowledge of the architecture of the rate processes, we want to retain as much generality as we can.

Renewal Theory counts the occurrence of repeated events whose evolution is described in terms of probability functions, usually functions of time. Each event is referred to as a renewal, and has a CDF that we shall designate as $P^{(n)}(t)$, where the parenthetical superscript indicates the number of the renewal. Events are sequential – that is, the first renewal occurs before the second, and so forth – and independent. The total number of renewals may be finite or infinite, although the latter situation is often assumed for the former to simplify the mathematics. We adopt this convention here, thus gaining a tacit assumption that the ending of the renewal process, for any of the reasons above, is smooth and has no effect on the dynamics of the process representation.

The expected number of renewals by time $t$ is expressed by the **Renewal Function**, which is defined as

$$P^{(R)}(t) \equiv \sum_{n=1}^{\infty} P^{(n)}(t), \tag{28}$$

where the parenthetical superscripted "R" indicates renewal. Because the CDF are assumed to be continuously differentiable, we may immediately define a **Renewal Density Function** (RDF) as

$$p^{(R)}(t) \equiv \sum_{n=1}^{\infty} p^{(n)}(t). \tag{29}$$

Some theorists extend the summation to $n = 0$ by defining a zeroth renewal event with CDF $P^{(0)}(t) \equiv 1$, from which we may deduce a PDF for this renewal of $p^{(0)}(t) = \delta(t)$, where $\delta(t)$ is a Dirac delta function.

As we have indicated, renewal theory describes repetitions of processes culminating in events, counting the events in a probabilistic manner.[1] These repetitions do not have to be

[1]Renewal theorists apparently do not like to talk about events. Since our interest is in (at least stochastically) observable events and the processes that produce these events are presumed to be repeated and the mathematics of renewal theory describes these processes and events well, we shall take the pragmatic but impure approach of continuing to use the verbiage of renewal theory.

identical and in many cases are not. We thus define the PDF of the $n^{th}$ renewal as $p_{(n)}(t)$, where the parenthetic subscript indicates the number of the renewal.[2] From this, we may write an evolution equation of a renewal in terms of the previous one, assuming the renewals are Markovian (that is, each renewal only depends on the previous one,) as

$$P^{(n)}(t) = \int_0^t G^{(n)}\left\{P^{(n-1)}, ..\right\}(t-t')\; p_{(n)}(t')\; dt' \tag{30}$$

where $G^{(n)}\{..\}$is the initiator of the initiation of the renewal; that is, it represents the probability that the process is initiated and may depend on the number of the renewal. Notice that this integral is symmetric in interchange of the time dependence of the two components. This will play an important role in subsequent developments. As a matter of convenience, we shall also write integrals of this form using the shorthand

$$P^{(n)} = G^{(n)} * p_{(n)}. \tag{31}$$

Two special situations are of note. First, the generator may be simple in form, depending only on the preceding renewal. In this case, Equation (30) has the form

$$P^{(n)}(t) = \int_0^t P^{(n-1)}(t-t')\; p_{(n)}(t')\; dt'. \tag{32}$$

We shall refer to this as a simple (or linear) renewal, although this terminology is somewhat different from that in the literature. We note that because Equation (32) is linear, we may immediately write the evolution equation of the corresponding renewal PDF as

$$p^{(n)}(t) = \int_0^t p^{(n-1)}(t-t')\; p_{(n)}(t')\; dt'. \tag{33}$$

Second, if all of the renewal PDFs are identical ($p_{(n)}(t) = p(t)$), then Equation (32) further reduces to

$$P^{(n)}(t) = \int_0^t P^{(n-1)}(t-t')\; p(t')\; dt', \tag{34}$$

with corresponding form of the PDF. We shall refer to this as a simple, uniform renewal.

Renewal Theory addresses the specific concept of rate through Blackwell's Theorem. This theorem states that

$$\lim_{t\to\infty} P^{(R)}(t+\Delta t) - P^{(R)}(t) \to \frac{\Delta t}{\mu}, \tag{35}$$

[2] Even in General Renewal Theory, all of the $p_{(n)}(t)$ are identical except the first. What we are doing here is a generalization of the mathematics of renewal theory. Our calling this renewal theory is loose, but handy. Our mathematical statement of renewal theory here is considerably more general than normally given. In most cases, our application of this theory will be considerably simpler and congruent with the usual statement.

where: $\mu \equiv \int_0^\infty t' \ p(t') \ dt'$ in the context of a simple, uniform renewal. The quantity $1/\mu$ is sometimes called the rate (of the renewal.) Within the definition of the renewal density function, this reduces to

$$\lim_{t\to\infty} p^{(R)}(t) \to \frac{1}{\mu}.$$

Recalling that Equation (30) is symmetric in interchange of the time dependence of the two functional pieces of the convolution, we may also write a differential equation form of the evolution equations as

$$\begin{aligned} \frac{d}{dt}P^{(n)}(t) &= G^{(n)}\left\{P^{(n-1)},..\right\}(t) \ p_{(n)}(0) \\ &\quad -\int_0^t G^{(n)}\left\{P^{(n-1)},..\right\}(t-t') \ \frac{d}{dt'}p_{(n)}(t') \ dt'. \end{aligned} \tag{36}$$

If the renewal PDF is NED with rate $\xi$, then Equation (36) further reduces to

$$\frac{d}{dt}P^{(n)}(t) = \xi G^{(n)}\left\{P^{(n-1)},..\right\}(t) \ - \xi P^{(n)}(t). \tag{37}$$

This is an exceedingly useful form.

Two further considerations are worthy of note here. First, the combination of a set of renewals is itself a renewal [Cox Smith 1954]. Second, renewals may be compounded.

**Remark 1.** Those who have had a course in renewal theory will likely be shaking their heads at this point. This is not the standard textbook renewal theory for a couple of quite good reasons. First of all, that textbook renewal theory isn't quite general enough to satisfy our needs for some of the problems that we want to solve. Second, the constraints that we need to apply are somewhat different from those normally applied in the textbook presentations. Yes, this is very different from vanilla renewal theory. It's different enough that I might have called it repetition theory except that then someone would have accused me of plagiarism.

### B. Order Statistics

By the postulates of General Rate Theory, we may deal with events that are caused by a collection of agents. The fact that we have a collection may mean that these agents interact in some fashion. In particular, we know that this collectivity has an effect on the probability of the event even when the agents act independently. The means for considering this is Order Statistics.

Where we used Basic Probability Theory and its extension Renewal Theory to describe the dynamics of events caused by a single element, we use **Order Statistics** to describe the behavior of the several as a group. The basic idea of Order Statistics is quite simple, but its implementation can be somewhat difficult, especially in terms of the basic calculations and computations that need to be performed. For this reason, it is useful to consider pictures.

The notion of pictures, as we use it here, is taken from physics, specifically quantum mechanics. In quantum mechanics, there are two pictures or interpretations of how the time dependence of the mathematical representation is viewed. The composition of quantum mechanics and indeed the nature of these pictures is not directly relevant here except for the fact that the two pictures are, within the restrictions of their formulations, equivalent and give identical observable results.

We want to consider two pictures of probability, which we may call the *implicit* and *explicit temporal representation* pictures. The explicit picture is the more familiar. In this picture, the probability functions are seen as dependent variables of (nominally) time. In the explicit time picture, we operate directly on the time variable. In the implicit picture, we operate directly on the probability functions as variables. This requires that the probability functions (at least the CDFs) are monotonically increasing over time and that there is a one-to-one correspondence between probability and time. This is not particularly difficult, but there is one idea that we must accept. In this picture, we may speak of the idea that a probability may have moments (for example, expected value) and because of the one-to-one relationship, this implies a time.

In particular, this also means that we must assume that the probability functions are invertible. That is, if $P(t)$ is the CDF that some (observable) event has occurred, then there exists an inversion $t(P)$ that is the unique time (not necessarily finite) when that probability value occurs.

We may now take up our consideration of Order Statistics.

Consider a collection of $N$ elements, each engaged in the same stochastic process (simple or complex) that results in an observable event. We thus expect to observe $N$ events, if we observe for a sufficiently long time. Assuming we know the stochastic mechanics of individual elements, given by a CDF $P(t)$, then we naturally ask the question of when do the events occur. (Or equivalently, What does the timing of a few events tell us about the collection?) The way we address this is by adopting the implicit time picture and back into the problem.

Let us specifically ask the question, What is the probability that the $r^{th}$event has occurred? If we write the unit function

$$\begin{aligned} Z &= [P + (1 - P)]^N \\ &\equiv 1, \end{aligned} \tag{38}$$

then a simple application of the binomial theorem allows us to write this function as a series

$$Z = \sum_{r=0}^{N} \binom{N}{r} (1 - P)^{N-r} \; P^r, \tag{39}$$

whose individual terms are the probabilities, within the collection environment, that the $r^{th}$event has occurred. Thus, for explicitness,

$$P_{r:N} = \binom{N}{r} (1 - P)^{N-r} P^r, \tag{40}$$

is the probability that the $r^{th}$event out of $N$ has occurred. Because the component probabilities are time dependent, this probability is also time dependent.

Since the CDF exists and is time dependent, this implies that its PDF, its first derivative with respect to time, exists as well. Taking this derivative is not completely straightforward. Because the term $(1 - P)^{N-r}$represents the probability that the event has not occurred for $N-r$ of the elements, it does not contribute to the derivative. Thus, the PDF of the $r^{th}$event is just

$$p_{r:N} = \binom{N}{r} r\,(1 - P)^{N-r} P^{r-1} p, \tag{41}$$

where $p \equiv \frac{d}{dt}P$. This is admittedly a very terse presentation and the student is referred to [David 1970] for details.

As before, this PDF is time dependent, and we may consider either the expected time the event occurs in the usual explicit time picture or the expected value of the individual element CDF in the implicit time picture. This demonstrates the value of the two pictures. The expected time of the event is

$$\begin{aligned} \bar{t}_{r:N} &= \int_0^\infty t\, p_{r:N}(t)\, dt \qquad (42) \\ &= \binom{N}{r} \int_0^\infty t\,(1 - P(t))^{N-r}\, rP(t)^{r-1}\, p(t)\, dt, \end{aligned}$$

which, given the explicit nature of the probability functions, is likely to be quite complicated. The expected probability of the event is

$$\begin{aligned} \overline{P}_{r:N} &= \int_0^1 P\, dP_{r:N}(P) \qquad (43) \\ &= \binom{N}{r} \int_0^1 P\,(1 - P)^{N-r}\, rP^{r-1} dP \\ &= \frac{r}{N+1}, \end{aligned}$$

which is considerably simpler. Of course, we must still compute the value of time, but this is given by the assumption of inversion of the CDF, $t\left(\frac{r}{N+1}\right)$. Note that we have made use of the differential relationship $p\,dt = dP$ and the integral expression $\int_0^1 x^{\mu-1}(1-x)^{\nu-1}\, dx = \frac{\Gamma(\mu)\,\Gamma(\nu)}{\Gamma(\mu+\nu)}$.

## VI. ON TO THE BASICS

The previous sections have been pretty heavy stuff, so let's cut through some of the fog. What we are all about is finding means for representing the counting of changes in populations of discrete events in a continuous manner. Our starting point for this is Equation (10),

$$\frac{d}{dx}f = \left[\frac{d}{dx}f\right] + \sum_i \Delta f_i \, \delta\left(x - \overline{x}_i\right),$$

which we shall refer to as the basic jump differential equation. In most cases, we are going to be dealing with situations where we start with discrete events and want to end up with a continuous representation of these events (as shown in Figure 1.) As a result, in general the continuous derivative term of this equation will be zero. Therefore, we are going to discard this term and reduce this equation to

$$\frac{d}{dx}f = \sum_i \Delta f_i \, \delta\left(x - \overline{x}_i\right). \tag{44}$$

We note that even should we have continuous derivative contributions, it is simple to put them back in, so we give up this modicum of generality with the understanding that we may re-insert it at any time, and by seizing this speciality, we gather some degree of simplicity and thereby clarity.

As a matter of course, we shall refer to the specific events as jumps because we presume them to be impulsive. These jumps are represented by the terms of the series on the right side of the above equations. We shall refer to the magnitudes of these jumps, $\Delta f_i$, as jump factors or jump functions. The positions of the jumps or events are the $\overline{x}_i$, which we commonly take to be times, but which may be spatial instead. Regardless, these jump positions or times are taken to be fundamentally independent of the jump differential equation although they may result from some phenomena that are described by some conjugate body of theory or experiment. To further clarify this, we rewrite this discrete event jump differential equation as

$$\frac{d}{dt}f = \sum_i \Delta f_i \, \delta\left(t - \bar{t}_i\right), \tag{45}$$

to explicitly show that we shall generally consider the jump positions to be times.

It is somewhat useful in developing this theory of rates to introduce a bit of taxonomy based on the nature of these jump factors/functions and times/positions. This taxonomy is summarized in the Table 1.

The taxonomy differentiates the situations of Deterministic and Stochastic behavior of the jump functions and times by numbered type. Thus, for example, a type one jump differential equation is characterized by both the jump function and the jump times being deterministic, while a jump differential equation of type four is characterized by both jump functions and jump times being stochastic. As we shall see, this taxonomy is presented largely to provide a framework for presenting methods for building the continuous forms of the jump differential equations from the discrete forms. The distinction being arbitrary, we are erecting no barriers among differential equations that may be of mixed type.

Table 1. Jump Taxonomy

| **Type** | **Jump Function Form** | **Jump Time Form** |
|---|---|---|
| 1 | Deterministic | Deterministic |
| 2 | Stochastic | Deterministic |
| 3 | Deterministic | Stochastic |
| 4 | Stochastic | Stochastic |

Our goal, as we have indicated, is to describe how we may start with discrete event differential equations, such as the above, and transform them into a continuous form. This taxonomy gives us a means of presenting different methods of transformation.

As a rule, in this presentation, we shall assume that the jump functions do not depend on the index of the jump; that is, they either depend only on time or are independent of the jump index. As a result, we may usually separate the jump function from the series.

## VII. TRANSFORMING THE DELTA FUNCTIONS

Let us now return to Equation (45), and as we have indicated, presume that the jump functions do not depend on the index of summation. In this case, we may rewrite the right side of the equation as

$$\sum_i \Delta f_i \, \delta\left(t - \bar{t}_i\right) = \Delta f\,(t) \sum_i \delta\left(t - \bar{t}_i\right), \tag{46}$$

where we have assumed that the jump function is a function of time. As we shall see shortly, this does not diminish the overall utility of the theory.

As we have also indicated, we want to find a means of transforming this discrete summation over events into some continuous representation. We now proceed to examine exactly this. Our discussion, for simplicity and clarity, is limited to a single or sequence of event times. Combinations of sets or sequences complicate the analysis but are ready extensions of the theory.

### A. Equally Spaced Deterministic Event Times

We start our presentation by considering a set of event times that are infinite in number and equally spaced. As a result, we may rewrite Equation (46) explicitly as

$$\Delta f\,(t) \sum_i \delta\left(t - \bar{t}_i\right) = \Delta f\,(t) \sum_{i=0}^{\infty} \delta\left(t - \bar{t}_i\right), \tag{47}$$

where we have arbitrarily assumed that the jumps begin at index zero. If we define $\bar{t}_i \equiv i \ \Delta t$, to assure that the events are evenly spaced in time, then this becomes

$$\Delta f(t) \sum_i \delta\left(t - \bar{t}_i\right) = \Delta f(t) \sum_{i=0}^{\infty} \delta(t - i \ \Delta t). \tag{48}$$

Let us now take the Laplace Transform of this equation

$$\begin{aligned} L\left[\Delta f(t) \sum_{i=0}^{\infty} \delta(t - i \ \Delta t)\right] &= \int_0^{\infty} e^{-st} \Delta f(t) \sum_{i=0}^{\infty} \delta(t - i \ \Delta t) dt, \\ &= \sum_{i=0}^{\infty} \int_0^{\infty} e^{-st} \Delta f(t) \ \delta(t - i \ \Delta t) dt, \\ &= \sum_{i=0}^{\infty} e^{-si \ \Delta t} \Delta f(i \ \Delta t), \end{aligned} \tag{49}$$

where we have had to pull the jump function back into the summation because it now depends on event number courtesy of the form of the delta functions. Now, if the jump function $\Delta f(t)$ is strictly polynomial (and preferably of low, positive order,) then this series is summable, although for even reasonably small powers of $i$, this summation will be difficult. Accordingly, we would prefer not to take this approach to making the transformation from discrete to continuous representation.

Accordingly, we limit ourselves to taking the Laplace Transform of just the series of delta functions

$$L\left[\sum_{i=0}^{\infty} \delta(t - i \ \Delta t)\right] = \sum_{i=0}^{\infty} e^{-si \ \Delta t}, \tag{50}$$

which we immediately see is summable as

$$L\left[\sum_{i=0}^{\infty} \delta(t - i \ \Delta t)\right] = \frac{1}{1 - e^{-s\Delta t}}, \tag{51}$$

since Equation (50) is a simple geometric series [Gradshteyn Ryzhik 1980, Section 1.112 Equation 1, p. 21.].

As it stands, this result is not an improvement and has gotten us no closer to a continuous representation. If however, we expand the exponential to first order,

$$e^{-s\Delta t} \simeq 1 - s\Delta t + HOT, \tag{52}$$

then we may approximate Equation (51) as

$$L\left[\sum_{i=0}^{\infty} \delta(t - i \ \Delta t)\right] \simeq \frac{1}{s\Delta t}, \tag{53}$$

which we may invert by inspection and obtain

$$\sum_{i=0}^{\infty} \delta\left(t - i\ \Delta t\right) \simeq \frac{1}{\Delta t}, \tag{54}$$

which is equivalent to an infinite time limit average (ala Blackwell's Theorem as we shall use subsequently in considering Type 3/4 jump differential equations.) That is, we spread the magnitude of the jumps over the entire duration of jumps.

This result transforms the jump differential equation to a continuous form,

$$\frac{d}{dt} f \simeq \frac{\Delta f\left(t\right)}{\Delta t}. \tag{55}$$

While this is a very useful result, it is limited in that we cannot extend the method to higher order approximations of the exponential without risking instability. Accordingly, let us now consider that instead of the first jump occurring at the origin, that it occurs after an increment of time. Thus, we rewrite Equation (47) as

$$\Delta f\left(t\right) \sum_{i} \delta\left(t - \bar{t}_i\right) = \Delta f\left(t\right) \sum_{i=1}^{\infty} \delta\left(t - \bar{t}_i\right). \tag{56}$$

We now follow the same method as before, assuming equal increment times, and obtaining

$$\begin{aligned} L\left[\sum_{i=1}^{\infty} \delta\left(t - i\ \Delta t\right)\right] &= \sum_{i=1}^{\infty} e^{-si\ \Delta t} \\ &= \frac{1}{1 - e^{-s\Delta t}} - 1 \\ &= \frac{e^{-s\Delta t}}{1 - e^{-s\Delta t}}, \\ &= \frac{1}{e^{s\Delta t} - 1}. \end{aligned} \tag{57}$$

This may be expanded as before. A first-order expansion gives us the same result as before, Equation (53). Unlike that expansion, which was limited to first order, we may expand Equation (57) to higher order. Thus, an expansion to second order gives us

$$L\left[\sum_{i=1}^{\infty} \delta\left(t - i\ \Delta t\right)\right] \simeq \frac{1}{s\Delta t + \frac{s^2 \Delta t^2}{2}}. \tag{58}$$

This can be separated using a partial fraction expansion, and inverted by inspection, giving us

$$\sum_{i=1}^{\infty} \delta\left(t - i\ \Delta t\right) \simeq \frac{1}{\Delta t}\left[1 - e^{-2t/\Delta t}\right]. \tag{59}$$

We immediately note that in the limit $t \to \infty$, this result becomes identical to Equation 54. Similarly, it gives us a continuous representation rate differential equation

$$\frac{d}{dt} f \simeq \frac{\Delta f\left(t\right)}{\Delta t}\left[1 - e^{-2t/\Delta t}\right]. \tag{60}$$

As a means of comparison, we plot the rate functions arising from the first three order expansions of the LT of the delta function series, Equation (57), in Figure 2. The third-order expansion is actually negative out to a time of about 0.7, but this is clipped in the plot. It may readily be seen that the detail of representation of the delta functions increases with expansion order. In a similar manner, we present the integrals of the first two order expansions versus the step trajectory in Figure 3. We may readily see that the type of "smoothing" we displayed earlier in Figure 1 occurs here, but we are able to see greater detail of the effects of expansion of the LT of the event delta function series. The first-order expansion provides a smooth representation that hits the vertical risers of the "steps," while the second order expansion hits at the mid points of the risers.

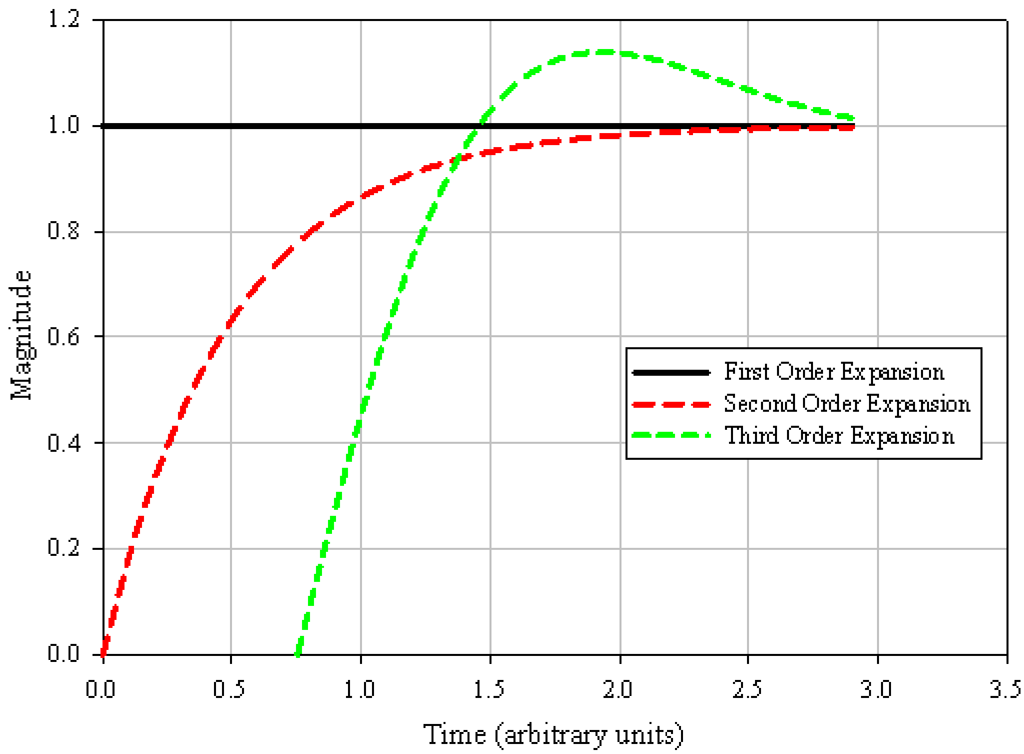


*Figure 2. Comparisons of First Three Order Expansions of Series of Event Delta Functions*

Before proceeding, it seems worthwhile to make a comparison of this method of developing the continuous representation by expansion of the Laplace Transform of the discrete representation with a more direct method. The direct method we shall consider is based on the expression of delta functions as Gaussians, Equation (4). As already noted, the Gaussian is a weak delta function, which means that in the limit $\sigma \to 0$, the Gaussian acts like a strong (or Dirac) delta function. Our approach here is the opposite of this. Instead of making $\sigma$ get smaller, we let it get larger. This is shown in Figure 4. As sigma is allowed to increase, the discrete Gaussians coalesce. Thus, we obtain the same type of continuous representation.

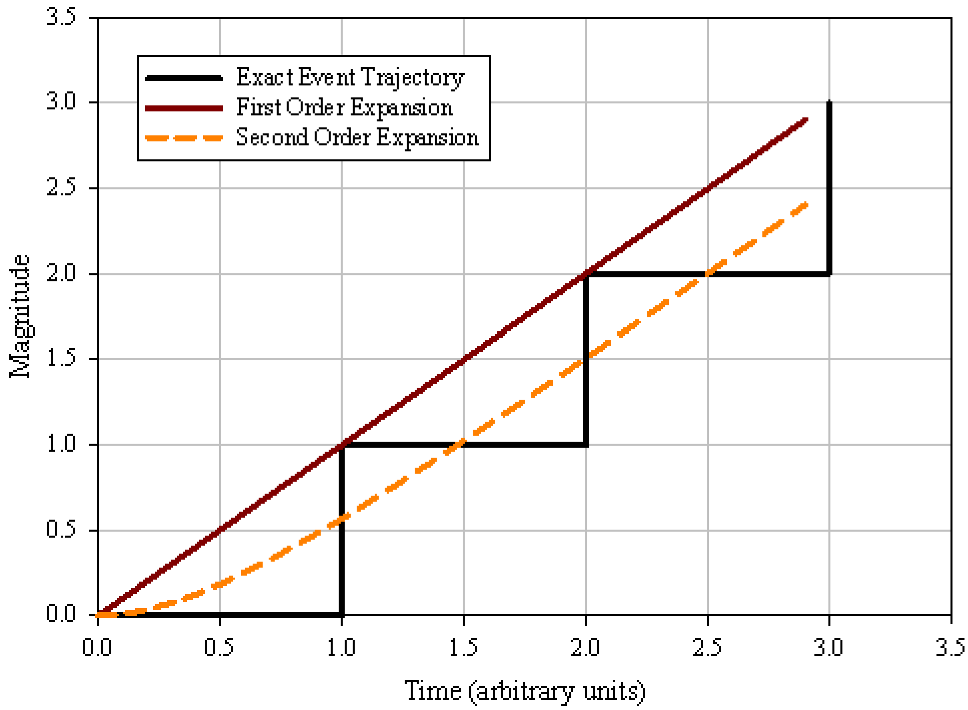


*Figure 3. Comparisons of Integrals of First Two Order Expansions of Series of Event Delta Functions With Actual Step Trajectory*

### B. Unequally Spaced Deterministic Event Times

The problem we encounter in dealing with unequally spaced[3] event times is that the transformation techniques used above to go from discrete to continuous representation are not generally applicable. To illustrate this, we return to Equation (56)

$$\Delta f\left(t\right)\sum_{i}\delta\left(t-\bar{t}_i\right)=\Delta f\left(t\right)\sum_{i=1}^{\infty}\delta\left(t-\bar{t}_i\right)\ ,$$

where we have assumed the jump function to be independent of index.

As a result, we need only consider the series of delta functions, whose Laplace Transform is

$$L\left[\sum_{i=1}^{\infty}\delta\left(t-\bar{t}_i\right)\right]=\sum_{i=1}^{\infty}e^{-s\bar{t}_i}. \tag{61}$$

By assumption there is no simple linear relationship between index and event times, and we cannot simply sum this series as we did before. If we examine the event times and

[3]Deterministic. By their nature, stochastic event times are unevenly spaced but are treated in a different manner.

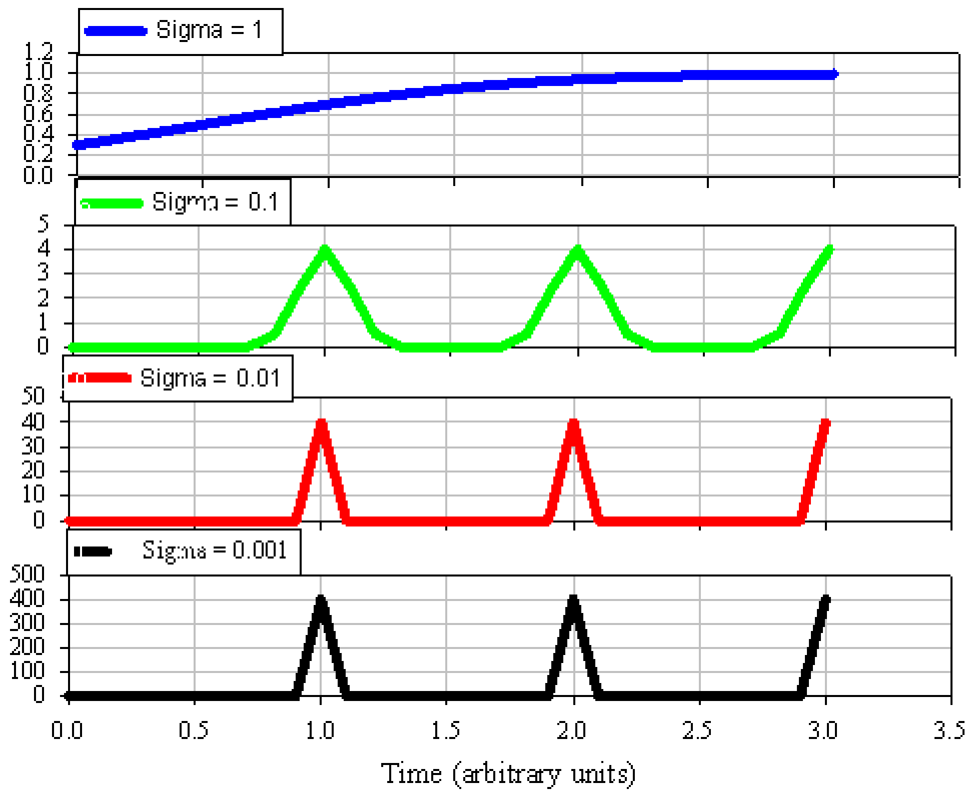


*Figure 4. Effect of Increasing Sigma*

find that the difference in adjoining event times is almost constant, then we may be able to approximate them as evenly spaced, or if the difference is almost constant except for a disordered but small amount, we might be able to approximate them as evenly spaced with a small stochastic component that can the transformed using the techniques described in the next subsection.

Generally however, we cannot sum Equation (61) in a simple manner so we turn rather quickly to transforming the series term by term using the techniques described in the previous subsection. Thus,

$$\begin{aligned}\sum_{i=1}^{\infty} e^{-s\bar{t}_i} = \sum_{i=1}^{\infty} & \frac{1}{e^{s\bar{t}_i}} \\ \simeq & \sum_{i=1}^{\infty} \frac{1}{1+s\bar{t}_i} \\ = & \sum_{i=1}^{\infty} \frac{1}{\bar{t}_i}\frac{1}{\frac{1}{\bar{t}_i}+s},\end{aligned} \tag{62}$$

which we may invert as

$$L^{-1}\left[\sum_{i=1}^{\infty} e^{-s\bar{t}_i}\right] \simeq \sum_{i=1}^{\infty} \frac{1}{\bar{t}_i} e^{-\frac{t}{\bar{t}_i}}. \tag{63}$$

Although elegant, this suffers from being tied to the origin. As a result, density of rate is often too bunched at short times. This is illustrated in Figure 5 where we compare this approximation with a set of uneven event times. The event times in this case are generated from the deciles of a Gaussian with mean and standard deviation selected to assure all the event times are nonnegative. The different heights of the delta functions are an artifact of the small standard deviation ($\sigma = 0.01$) and the calculation increment. The general failure of the approximation to capture the behavior of the events is obvious in this case.

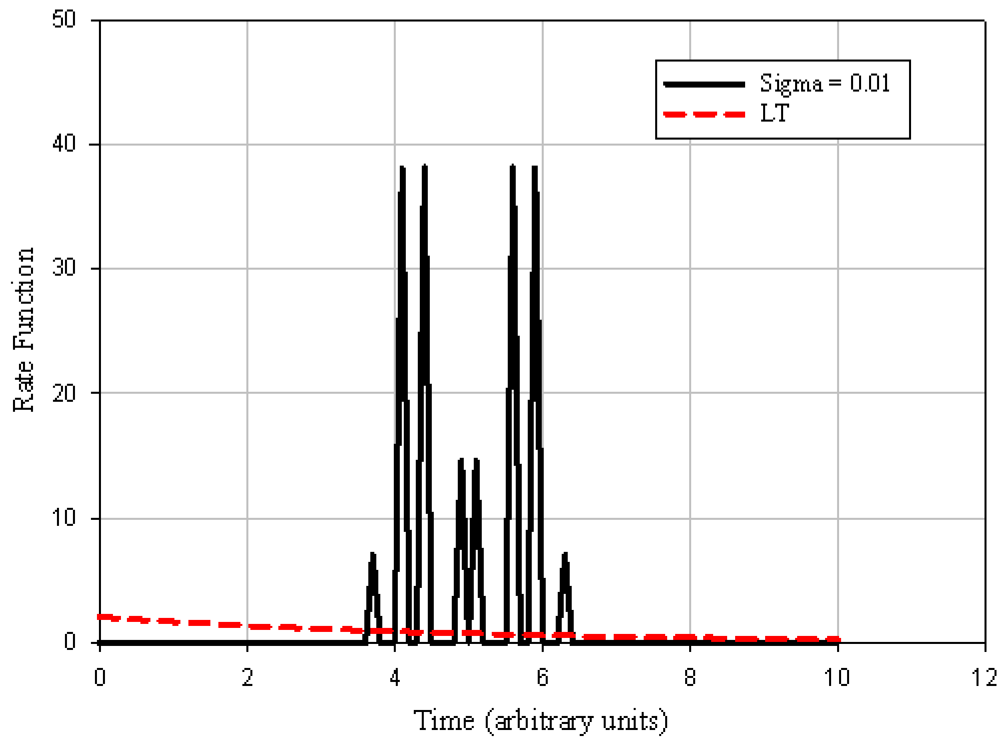


*Figure 5. Comparison of Uneven Event Times Delta Functions And Term-By-Term LT Expansion*

As a result, we are often forced on the more difficult task of taking some representation of the delta function, relaxing its strength (increasing its weakness), and looking for some pattern of behavior that can be exploited in forming a continuous (approximate) representation. For example, if we take the same event times as used in Figure 5, represent the delta functions as Gaussians (which we did in that figure for plotting purposes anyway,) and allow the standard deviation ($\sigma$) to increase, then we find a pattern coalescing. This is shown in Figure 6.

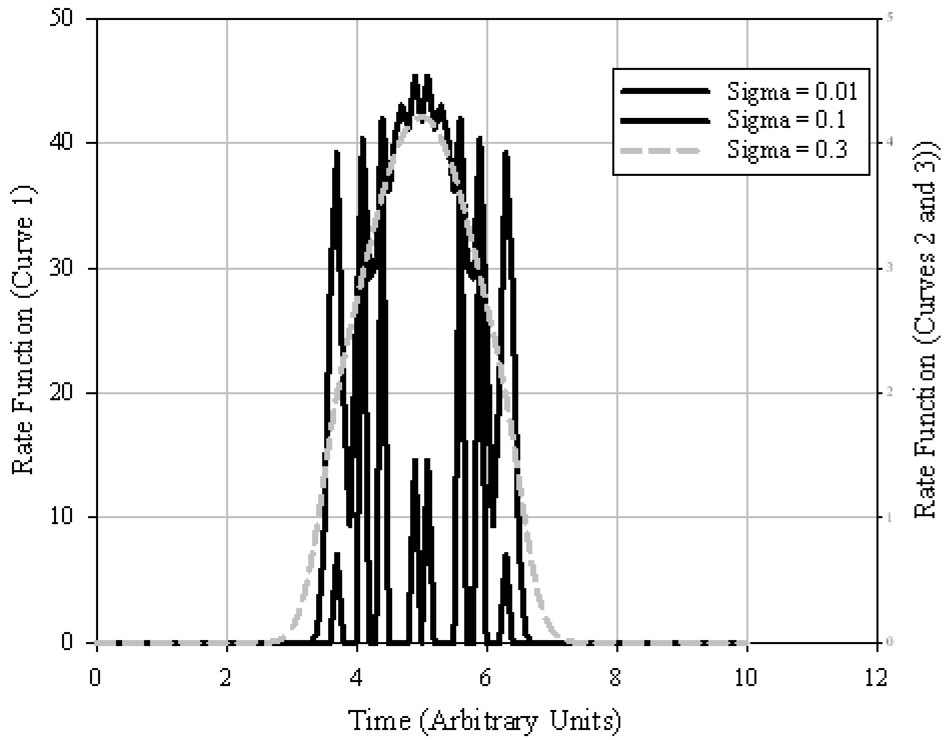


*Figure 6. Comparison of Uneven Event Times as Gaussians of Different Standard Deviations*

We first note that to avoid hiding details because of maximum amplitude changes, we have plotted the same representation of the delta functions of Figure 5 on one vertical axis and the other two representations on a second vertical axis. The amplitude variation of the small standard deviation representation is again an artifact of calculation. When we relax the standard deviation from 0.01 to 0.1, then we see the emergence of an envelope with the peaks sticking through. If we further relax the standard deviation to 0.3, then the curve smooths and we recognize it as a Gaussian, which is what we should expect given the event times being the deciles of a Gaussian.

### C. Stochastic Event Times

As indicated, the times of deterministic events are represented by strong (Dirac) delta functions and we form the continuous representation by weakening them in some manner. These need no further consideration at this point. The behavior of the stochastic events does require development, however. Being stochastic, we intuit that the event times are represented by delta functions that are inherently weak in some sense. To examine this, we rewrite our jump differential equation, Equation (44), with probability as the independent variable. Thus,

$$df\left(t\right)=\sum_{i}\Delta f_{i}\,\delta\left(P-\overline{P}_{i}\right)\,dP\left(t\right). \tag{64}$$

This telegraphs that we are going to use the implicit temporal representation picture, at least initially.

Now consider a collection of $N$ elements. These elements cause events that define the solution $f\left(t\right)$ of the differential equation. We start with the restriction that each element causes (at most) one event. Under this restriction, basic probability theory defines the CDF-PDF of the event caused by an individual element. Thus, $P\left(t\right)$ represents the cumulative probability that the event caused by an individual element has occurred.

Because the elements are parts of a collective, the individual events are described by order statistics. As a result, the PDF of the $r^{th}$ of $N$ events is given by Equation (41), which we now write, with differential, as

$$p_{r:N}\left(t\right)dt=\binom{N}{r}\left(1-P\right)^{N-r}rP^{r-1}dP. \tag{65}$$

It may be noted that the left side is explicitly a function of time and the right side is explicitly a function of individual element probability. The right side may be considered to be the PDF as a function of probability (times a differential). We concentrate on the non-differential portion. This is

$$p_{r:N}\left(P\right)=\binom{N}{r}\left(1-P\right)^{N-r}rP^{r-1}. \tag{66}$$

Let us now examine this function of the individual element CDF. We plot selected functions in Figure 7. These functions clearly have the appearance of weak delta functions in the sense of our earlier discussion, but this needs to be demonstrated in the context of Equation (5). To do this, we return to Equation (66), and rewrite it somewhat as

$$p_{r:N}\left(t\right)dt=\binom{N}{r}\left(1-P\right)^{N-r+1}rP^{r}d\ln\left(\frac{P}{1-P}\right). \tag{67}$$

We select the variable $y=\ln\left(\frac{P}{1-P}\right)$ for two reasons. First, it gives us natural limits of $\pm\infty$, and second, as we shall see, it reproduces the expected values of Equation (41). Removing the differential, we write the quantity as

$$p_{r:N}\left(\ln\left(\frac{P}{1-P}\right)\right)=\binom{N}{r}re^{(N-r+1)\ln(1-P)+r\ln(P)}. \tag{68}$$

Since we know these functions, at least in the P picture, are peaked, we may expand the exponent in a McWheeny or (pseudo) Saddle Point Expansion. This is just a Taylor series expansion where the expansion point is the extremum, normally the maximum. By this, we mean that we expand the exponent in $y = \ln\left(\frac{P}{1-P}\right)$ and set the first derivative to zero. This determines the expansion point. For simplicity of presentation, we are only going to keep the first two derivative terms, recognizing that this results in some error, but we only want to demonstrate the weak delta function behavior of this class of functions.

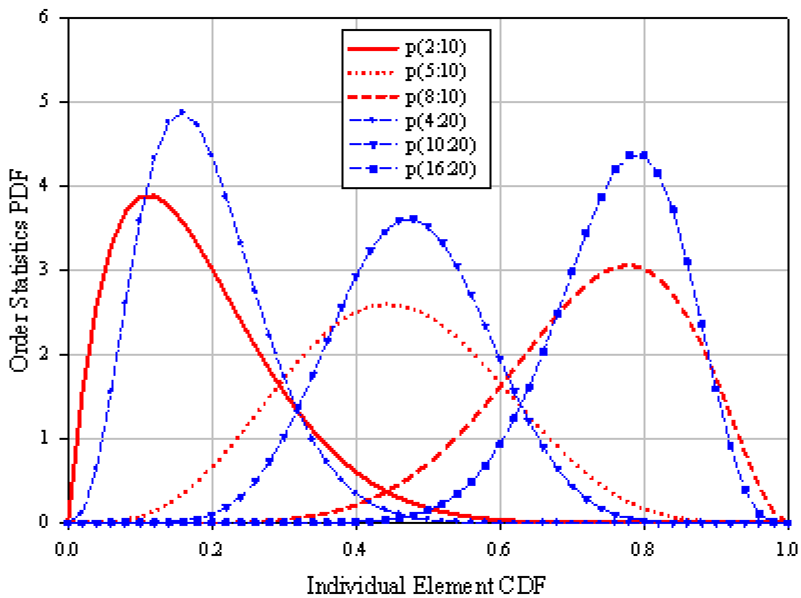


*Figure 7. Order Statistics PDF Examples*

We now write out the expansion explicitly to second order,

$$\begin{aligned}
F &= (N-r+1)\ln(1-P) + r\ln(P), \\
\frac{d}{dy}F &= \frac{\partial}{\partial \ln\left(\frac{P}{1-P}\right)}F \\
&= \frac{dP}{d\ln\left(\frac{P}{1-P}\right)}\frac{d}{dP}F; \\
&= P(1-P)\left[(N-r+1)\frac{-1}{1-P} + \frac{r}{P}\right] \\
&= -(N+1)P + r; \\
\frac{d^2}{dy^2}F &= \frac{\partial}{\partial \ln\left(\frac{P}{1-P}\right)}\left(\frac{d}{dy}F\right) = \frac{dP}{d\ln\left(\frac{P}{1-P}\right)}\frac{d}{dP}\left(\frac{d}{dy}F\right) \\
&= -(N+1)P(1-P).
\end{aligned} \tag{69}$$

The third and fourth derivatives are

$$\begin{aligned}
\frac{d^3}{dy^3}F &= -(N+1)\,P\,(1-P)\,(1-2P)\,, \qquad (70)\\
\frac{d^4}{dy^4}F &= -(N+1)\,P\,(1-P)\left(1-6P+6P^2\right).
\end{aligned}$$

The pattern is not immediately obvious, but at some point the derivative changes sign (depending on the value of $P$) and thereafter becomes oscillatory. This series thus also becomes uniformly convergent and assures that the behavior we are capturing in this two-term expansion is trustworthy.

If we now require the first derivative to be zero, we obtain the expansion point $P = \frac{r}{N+1}$, which is just the expected value of the probability, Equation (43). Substitute this into Equation (70),

$$\begin{aligned}
F &= (N-r+1)\ln\left(1-\frac{r}{N+1}\right) + r\ln\left(\frac{r}{N+1}\right), \qquad (71)\\
&= (N-r+1)\ln\left(\frac{N-r+1}{N+1}\right) + r\ln\left(\frac{r}{N+1}\right);\\
\frac{d}{dy}F &= -(N+1)\,\frac{r}{N+1} + r,\\
&= 0;\\
\frac{d^2}{dy^2}F &= -(N+1)\,\frac{r}{N+1}\,\frac{N-r+1}{N+1},\\
&= -\frac{r\,(N-r+1)}{N+1},
\end{aligned}$$

which gives us the expansion

$$\begin{aligned}
F(y) \simeq\ & (N-r+1)\ln\left(\frac{N-r+1}{N+1}\right) + r\ln\left(\frac{r}{N+1}\right) \qquad (72)\\
& -\frac{r\,(N-r+1)}{N+1}\,\frac{\left[y-\ln\left(\frac{r}{N-r+1}\right)\right]^2}{2} + O\left(y^3\right).
\end{aligned}$$

If we now define $\overline{y} \equiv \ln\left(\frac{r}{N-r+1}\right)$, $\sigma^2 \equiv \frac{N+1}{r(N-r+1)}$, then we may rewrite Equation (72) as

$$f(x) \simeq (N-r+1)\ln\left(\frac{N-r+1}{N+1}\right) + r\ln\left(\frac{r}{N+1}\right) - \frac{(y-\overline{y})^2}{2\sigma^2}, \qquad (73)$$

and thus rewrite Equation (68) as

$$p_{r:N}(y) = \binom{N}{r} r e^{(N-r+1)\ln\left(\frac{N-r+1}{N+1}\right)+r\ln\left(\frac{r}{N+1}\right)-\frac{(y-\overline{y})^2}{2\sigma^2}}. \qquad (74)$$

This equation may be more explicitly rewritten and rearranged as

$$p_{r:N}(y) = \frac{N!\,r}{r!\,(N-r)!}\frac{(N-r+1)^{N-r+1}\,r^r}{(N+1)^{N+1}}e^{-\frac{(y-\overline{y})^2}{2\sigma^2}}. \tag{75}$$

To make factors cancel correctly, we further rewrite this as

$$p_{r:N}(y) \simeq \frac{(N+1)!\,r\,(N-r+1)}{(N+1)\,r!\,(N-r+1)!}\frac{(N-r+1)^{N-r+1}\,r^r}{(N+1)^{N+1}}e^{-\frac{(y-\overline{y})^2}{2\sigma^2}}, \tag{76}$$

and make use of Stirling's Approximation for the factorial

$$n! \simeq n^{n+1/2}e^{-n}\sqrt{2\pi}. \tag{77}$$

Now consider just the terms on the right side of Equation (76) leading the exponential,

$$\begin{aligned}
&\frac{(N+1)!\,r\,(N-r+1)}{(N+1)\,r!\,(N-r+1)!}\frac{(N-r+1)^{N-r+1}\,r^r}{(N+1)^{N+1}} \qquad (78)\\
\simeq\;& \frac{(N+1)^{N+1+1/2}\;e^{-N-1}r\,(N-r+1)}{\sqrt{2\pi}\,(N+1)\,r^{r+1/2}e^{-r}\,(N-r+1)^{N-r+1+1/2}\,e^{-N+r-1}}\frac{(N-r+1)^{N-r+1}\,r^r}{(N+1)^{N+1}}\\
=\;& \frac{(N+1)^{1/2}\;e^{-N-1}r\,(N-r+1)}{\sqrt{2\pi}\,(N+1)\,r^{1/2}e^{-r}\,(N-r+1)^{1/2}\,e^{-N+r-1}}\\
=\;& \frac{(N+1)^{1/2}\;r\,(N-r+1)}{\sqrt{2\pi}\,(N+1)\,r^{1/2}\,(N-r+1)^{1/2}}\\
=\;& \frac{1}{\sqrt{2\pi}}\sqrt{\frac{r\,(N-r+1)}{N+1}}\\
=\;& \frac{1}{\sqrt{2\pi}\sigma}.
\end{aligned}$$

This reduces Equation (76) to

$$p_{r:N}(y) \simeq \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(y-\overline{y})^2}{2\sigma^2}}, \tag{79}$$

which we immediately recognize as a Gaussian. The standard deviation $\sigma \sim O\left(\frac{1}{\sqrt{N}}\right)$ so that $\sigma \to 0$ weakly as $N \to \infty$. This satisfies the definition of a weak delta function, and we may thus rewrite Equation (68) as

$$p_{r:N}(t)\,dt = \delta(y-\overline{y})\,dy, \tag{80}$$

in this weak sense. The relationship of the Saddle Point Expansion to the original functions is shown in Figure 8. The agreement is inexact because we truncated the expansion. It is also worthwhile to note that this expansion is valid for $N > 0$. This is important since it allows us to address collections of cause agents of population one.

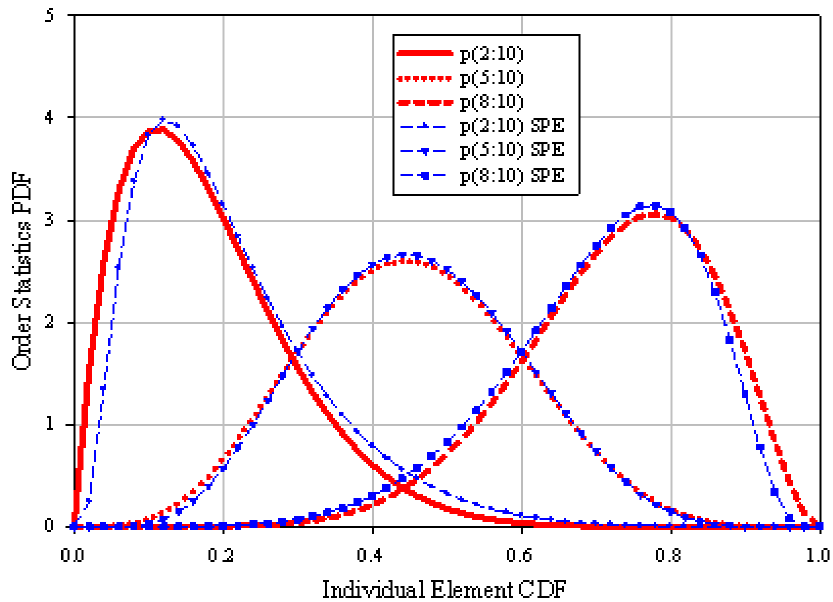


*Figure 8. Order Statistics PDFs and Their Saddle Point Expansions*

## VIII. RATE DIFFERENTIAL EQUATIONS

We now complete the derivation of the stochastic rate differential equations, review the deterministic rate differential equations for evenly space event times, and compare the two.

### A. Stochastic Rate Differential Equations

We now combine this result with our jump differential Equation (64),

$$df\,(x) \quad = \quad \sum_i \Delta f_i\, \delta\,(x - \overline{x}_i)\; dx, \tag{81}$$

$$\Longrightarrow \quad df\,(t) = \sum_{r=1}^{N} \Delta f_r\, p_{r:N}\,(t)\, dt.$$

The differential equation is now a function of time, and the summation over jumps has become explicit. This is the basic stochastic rate differential equation (except for the piecewise continuous derivative term.) Although, we do not, at this point in the development, know the explicit form of the jump function, some further elaborations are still possible.

When the jump function is not explicitly a function of $i$, we may pull the jump function out of the summation. Further, if we substitute the exact expression for the order statistics PDF, Equation (65), we obtain

$$df\,(t) = \Delta f\,(t) \sum_{r=1}^{N} \binom{N}{r} (1-P)^{N-r}\, r P^{r-1} p\, dt. \tag{82}$$

Writing out the binomial coefficient and canceling the $r$ term yields

$$df\left(t\right)=\Delta f\left(t\right)\sum_{r=1}^{N}\frac{N!}{\left(r-1\right)!\left(N-r\right)!}\left(1-P\right)^{N-r}P^{r-1}p\,dt. \tag{83}$$

We now make the change of summation index $s=r-1$, so that Equation (83) becomes

$$\begin{aligned} df\left(t\right) &= \Delta f\left(t\right)\sum_{s=0}^{N-1}\frac{N!}{s!\left(N-s-1\right)!}\left(1-P\right)^{N-s-1}P^{s}p\,dt \\ &= \Delta f\left(t\right)Np\sum_{s=0}^{N-1}\frac{\left(N-1\right)!}{s!\left(N-s-1\right)!}\left(1-P\right)^{N-s-1}P^{s}\,dt \\ &= \Delta f\left(t\right)Np\,dt. \end{aligned} \tag{84}$$

This collapse results from recognition that the middle representation of Equation (84) is itself a Binomial Expansion. If we now recognize, based on our earlier development of the real-valued binomial expansion, that $N$ may be time dependent and that the individual element PDF is time dependent, we may write this as

$$\frac{d}{dt}f\left(t\right)=\Delta f\left(t\right)\,p\left(t\right)\,N\left(t\right). \tag{85}$$

This is a general result for the case when the jump function does not depend on the event number.

Before concluding this section, two other cases need to be considered. The first is when the jump process is repeated and described by renewal theory. In this case, we make use of the independence of the individual renewals and extend it to the order statistics. Equation (81) becomes

$$df\left(t\right)=\sum_{n=1}^{\infty}\sum_{r=1}^{N}\Delta f_{r}^{(n)}\,p_{r:N}^{(n)}\left(t\right)dt, \tag{86}$$

where $\Delta f_r^{(n)}$ and $p_{r:N}^{(n)}\left(t\right)$ are now the jump function and order statistic PDF, respectively of the $r^{th}$ event of $N$ of the $n^{th}$ renewal or repetition. If the jump function does not depend on event number, but does depend on renewal number, then this equation may be written as

$$df\left(t\right)=\sum_{n=1}^{\infty}\Delta f^{(n)}\sum_{r=1}^{N}p_{r:N}^{(n)}\left(t\right)dt. \tag{87}$$

This equation collapses as before to give

$$df\left(t\right)=\sum_{n=1}^{\infty}\Delta f^{(n)}p^{(n)}N\,dt. \tag{88}$$

If the jump function depends on neither event nor renewal number, then this becomes

$$df\left(t\right)=\Delta f\left(t\right)\sum_{n=1}^{\infty}p^{(n)}N\,dt, \tag{89}$$

which reduces to

$$df\left(t\right)=\Delta f\left(t\right)p^{(R)}N\,dt. \tag{90}$$

This is the form that we shall use subsequently for developing many particular rate differential equations. It should be noted that this is a special case of the more general result.

The second case that needs to be developed is for $N=1$. As previously noted, the (pseudo) Saddle Point Expansion is valid even for this limiting case, so we may return to Equation (85) and rewrite it as

$$\begin{aligned} df\left(t\right) &= \Delta f_1\,p_{1:1}\left(t\right)dt, \\ &= \Delta f\,p\left(t\right)dt. \end{aligned} \tag{91}$$

We note immediately that the jump function is not dependent on event number because there is only one event. If there are multiple events, however, and the jump process is repeated, this differential equation becomes

$$df\left(t\right)=\sum_{n=1}^{\infty}\Delta f^{(n)}\,p^{(n)}\left(t\right)\,dt. \tag{92}$$

Because we will be making a transformation from a discrete view to a continuous view, the values of our trajectories will be transformed from integer valued to non-integer (nominally real) valued. It is thus useful to examine the behavior of the order statistics CDFs when the number of elements in the collection is non-integer.

## B. Extension to Non-Integer Collections

As already discussed, Equation (40), the probability that the $r^{th}$event among $N$ is given by a simple term from a binomial expansion,

$$P_{r:N}\left(t\right)=\binom{N}{r}P\left(t\right)^{r}\,\left[1-P\left(t\right)\right]^{N-r}, \tag{93}$$

where $P\left(t\right)$ is the probability that the event has occurred by time $t$ for an individual element in isolation. These probabilities are inherently normalized,

$$\begin{aligned} \sum_{r=0}^{N}P_{r:N}\left(t\right) &= \sum_{r=0}^{N}\binom{N}{r}P\left(t\right)^{r}\,\left[1-P\left(t\right)\right]^{N-r} \\ &= \left[P\left(t\right)+\left[1-P\left(t\right)\right]\right]^{N} \\ &= 1, \end{aligned} \tag{94}$$

from their basic formulation, so this expression is a bit of a tautology. It means, however, that so long as $N$ has integer value, the collection of individual event probabilities is normalized for all values of time regardless of any evolution.

At this point, we lean forward and anticipate ourselves. Specifically, we want to examine what happens when we allow $N$ to have non-integer (real) values. We may proceed by considering the function

$$\begin{aligned} Z_R &= [P(t) + [1 - P(t)]]^R, \\ &= 1, \end{aligned} \tag{95}$$

which is the real valued equivalent of Equation (38). The exponent $R$ is explicitly non-integer. For convenience, we adopt the notation

$$\begin{aligned} x &\equiv P(t), \\ y &\equiv [1 - P(t)], \end{aligned} \tag{96}$$

so that we may write

$$Z_R = [x + y]^R. \tag{97}$$

This may appear to be an excessive distinction, but it may be recalled that this is exactly the same type of procedure we used in calculating the derivative above. It is also a common and useful distinction in manipulating these expressions [Mohling 1982].

We expand this function in a Maclaurin series about $x = 0$, giving us (in a naive textbook fashion),

$$Z_R = \sum_{k=0}^{\infty} \left. \frac{\partial^k Z_R}{\partial x^k} \right|_{x=0} \frac{x^k}{k!}. \tag{98}$$

The derivatives may be similarly computed as

$$\left. \frac{\partial^k Z_R}{\partial x^k} \right|_{x=0} = \left[ \prod_{l=0}^{k-1} (R - l) \right] y^{R-k}. \tag{99}$$

We note immediately that for values of $k > [R]$, where [] indicates the integer value of the argument, the exponent of $y$ becomes negative, and the sign of the product begins to oscillate. The series, Equation (98), thus becomes oscillatory beyond this value of $k$. Further, for this part of the series, the product will grow in magnitude slower than $k!$, so that each succeeding term not only oscillates, but also gets smaller.

This series, Equation (98), is thus Uniformly Convergent and may be adequately approximated as

$$Z_R \simeq \sum_{k=0}^{[R]} \left. \frac{\partial^k Z_R}{\partial x^k} \right|_{x=0} \frac{x^k}{k!}, \tag{100}$$

since the higher terms are oscillatory and largely cancel. If we make use of the Gamma function, we may rewrite this as

$$Z_R \simeq \sum_{k=0}^{[R]} \frac{\Gamma(R+1)}{\Gamma(R+1-k)} y^{R-k} \frac{x^k}{k!}. \tag{101}$$

This is the real number equivalent of the Binomial Theorem. In a like manner to the previous, we may write the PDF of the $k^{th}$ event as

$$p_{k:R} = \frac{\Gamma(R+1)}{\Gamma(R+1-k)} y^{R-k} \frac{x^{k-1}}{(k-1)!}, \tag{102}$$

by exactly the same type of development as led us to Equation (41). From this, we may make the following observations:

1. The expected value of the probability of the $k^{th}$ of $R$ events is well defined in the same manner as Equation (43);
2. The delta function behavior of the PDF of the $k^{th}$ of $R$ events is well defined in the same manner as Equations (65 through 80); and
3. The summation of the PDF of the $k^{th}$ of $R$ events, $k = 1..[R]$ (actually $\infty$), is well defined and identical in form to Equations (82 through 85). This latter follows from the derivation of Stirling's Approximation from the Gamma function with the factorial as a special case.

Thus, this approximation not only has components that look like the purely integer index form but also has comparable expansion properties. Based on these observations, the behavior of the "Order Statistics" for non-integer and integer populations is sufficiently close that we may treat them identically. The results derived above are happily applicable to real as well as integer numbers.

### C. Deterministic Rate Equations

It may be recalled that meaningful general transformations could only be developed for evenly spaced deterministic event times. We limit our immediate discussion to such, noting that if the event times are suitably disordered, or have some pattern, then they may be represented by means akin to those just developed for stochastic events.

For single sequences (sets) of evenly spaced deterministic event times, where the event jump functions are independent of index, we identify two cases. In the first case, events begin in index zero, that is, at the temporal origin, and give rise to a rate differential equation,

$$\frac{d}{dt} f \simeq \frac{\Delta f(t)}{\Delta t}, \tag{103}$$

where: $\Delta f(t) =$ the jump function, and $1/\Delta t =$ the continuous representation (approximation) of the jumps that occur with interval $\Delta t$. In the second case, the jump events begin with index one, and give rise to a rate differential equation,

$$\frac{d}{dt} f \simeq \frac{\Delta f(t)}{\Delta t} \left[1 - e^{-2t/\Delta t}\right]. \tag{104}$$

We note that the Equation (104) asymptotically reduces to the form of Equation (103) as time increases.

### D. Comparison of Deterministic and Stochastic Rate Differential Equations

We have already noted that for stochastic events, the rate differential equation takes the form

$$\frac{d}{dt}f(t) = \Delta f(t)\ p(t)\ N(t), \tag{105}$$

when the events (jumps) are not repetitive and

$$\frac{d}{dt}f(t) = \Delta f(t)\ p^{(R)}(t)\ N(t), \tag{106}$$

when the events are repetitive. In some cases, deterministic and stochastic rate differential equations may take the same form.

The first case of this is the most obvious. If a stochastic process is uniformly distributed on $[t_1, t_2]$, then $p(t) = 1/(t_2 - t_1)$, $t_1 \le t \le t_2$, and Equation (105) reduces to

$$\frac{d}{dt}f(t) = \frac{\Delta f(t)}{t_2 - t_1} N(t), t_1 \le t \le t_2, \tag{107}$$

which is of the same form as Equation (103).

Next, if a repeated process (described by Renewal Theory) is uniform and simple with NED distribution $p(t) = \alpha\ \exp(-\alpha t), \alpha = 1/\langle t \rangle, \langle t \rangle = \int_0^\infty t'\ p(t')\ dt'$, then $p^{(R)}(t) = \alpha$, and this reduces Equation (106) to

$$\frac{d}{dt}f(t) = \frac{\Delta f(t)}{\langle t \rangle} N(t), \tag{108}$$

which is also identical in form to Equation (103). Thus, in both of these cases, we have stochastic processes with rate differential equations that are identical to deterministic rate differential equations.

Lastly, consider a repeated process that is alternating in two NED, $p(t) = \alpha\ \exp(-\alpha t)$, and $q(t) = \beta\ \exp(-\beta t)$. The renewal density function of this process is

$$p^{(R)}(t) = \frac{\alpha\beta}{\alpha+\beta}\left[1 - e^{-(\alpha+\beta)t}\right]. \tag{109}$$

This allows us to write Equation (106) as

$$\frac{d}{dt}f(t) = \frac{\alpha\beta}{\alpha+\beta}\Delta f(t)\ N(t)\left[1 - e^{-(\alpha+\beta)t}\right], \tag{110}$$

which is of the same form as Equation (104).

While NED renewals are essentially the simplest that may be formed, they are nonetheless quite common because of the frequency of occurrence of NED processes in nature. Regardless, we must acknowledge that the form of a rate differential equation does not uniquely indicate that the process it represents is either stochastic or deterministic.

## IX. DETERMINISTIC EXAMPLES

We now essentially change our approach from theory to example. The theoretical basis of Rate Theory having been established, it now becomes useful to consider several examples of how Rate Theory may be applied to the development of Rate Differential Equations. In this first section of examples, we concentrate on examples of deterministic rate processes, those of Type 1.

### A. Pong

Our first example is based on the computer game Pong. This game, one of the earliest, is actually a very complicated mechanics problem. In essence, it is the Classical (as opposed to Quantum) Mechanical one-dimensional particle-in-the-box problem.

The problem consists of the following: a one-dimensional box of dimension $d$ contains a single particle moving with speed of magnitude $u_0$. The box is assumed to have infinitely more mass than the particle. Collisions between box walls are elastic and lossless.

As a result, both energy and momentum are conserved. The particle moves in a closed trajectory (an orbit on $[0, d]$) between the two walls with speed (velocity) of $\pm u_0$. Normally, we should not apply Rate Theory to mechanics problems since the formalisms of Mechanics; Newton's equations; or the derivative operations of the Lagrangian, Routhian, or Hamiltonian functions; provide a direct means of developing the differential equations of mechanics [Goldstein 1950]. However, because this problem has nonholonomic constraints, it is useful to develop a Rate Theory example, approximate though it is, to show a special application of the theory.

Despite the nonholonomic constraint, this problem is so simple that an algorithmic solution can be written trivially. If we define the transit time of the particle from one side of the box to another as $\Delta t$, then we have

$$\Delta t = \frac{d}{u_0}. \tag{111}$$

It is then useful to define the integer $j^*$ as

$$j^* = \text{integer}\left(\frac{u_0 t}{d}\right), \tag{112}$$

where $t =$ the time since the particle satisfied the conditions $x(0) = 0, u(0) = u_0$, where $x(t)$, $u(t)$ are, respectively, the position and speed of the particle at time $t$. These are what we might consider to be initial conditions. In this context, $j^*$ is just the number of times that the particle has bounced off the walls since the initial time.

From this we may write that

$$x(t) = \begin{cases} u_0 (t - j^* \Delta t), & j^* \ even \\ d - u_0 (t - j^* \Delta t), & j^* \ odd \end{cases}, \tag{113}$$

which is indeed the solution of the motion of the particle, albeit a piecewise solution.

If we return to Equation (44),

$$\frac{d}{dx} f = \sum_i \Delta f_i \, \delta (x - \overline{x}_i),$$

then we may write the rate differential equation of the particle's speed as

$$\frac{d}{dt} u = \sum_j \Delta u_j \, \delta \left(t - \bar{t}_j\right). \tag{114}$$

We have switched indices from $i$ to $j$ for reasons that will be abundantly clear shortly.

The jump times $\bar{t}_j$ are the times when the particle collides with one or other of the walls, which is just

$$\bar{t}_j = j \Delta t. \tag{115}$$

When the collision occurs, the speed of the particle switches from positive to negative, so we may write the jump function in a still general form as

$$\Delta u_j = (-1)^j \, \Delta u, \tag{116}$$

where we have left the magnitude of the jump unspecified to allow us to correct for the approximation. This is a case where the jump function does depend on index but, as we shall see, is simple enough (that is, linear) that we may treat it in closed form.

Before proceeding, it is useful to recall the identity

$$-1 = e^{\pm \pi i}, \tag{117}$$

where $i \equiv \sqrt{-1}$, which illuminates our need to change index notation. This identity allows us to rewrite Equation (116) as

$$\Delta u_j = e^{\pi i j} \Delta u, \tag{118}$$

which in turn allows us to write the jump terms as

$$\begin{aligned} \sum_j \Delta u_j \, \delta \left(t - \bar{t}_j\right) &= \sum_{j=1}^{\infty} e^{\pm \pi i j} \Delta u \, \delta (t - j \Delta t), \\ &= \Delta u \sum_{j=1}^{\infty} e^{\pm \pi i j} \, \delta (t - j \Delta t). \end{aligned} \tag{119}$$

We may now take the Laplace Transform of the right side of Equation (119) as

$$L\left[\Delta u \sum_{j=1}^{\infty} e^{\pi ij}\, \delta\left(t - j\Delta t\right)\right] = \Delta u \sum_{j=1}^{\infty} e^{-sj\Delta t} e^{\pm \pi ij}, \tag{120}$$

This series is exactly summable, yielding

$$\sum_{j=1}^{\infty} e^{-sj\Delta t} e^{\pm \pi ij} = \frac{1}{e^{s\Delta t \mp \pi i} - 1}, \tag{121}$$

which we now expand to first order using the techniques described previously for evenly spaced event times as

$$\sum_{j=1}^{\infty} e^{-sj\Delta t} e^{\pm \pi ij} \simeq \frac{1}{s\Delta t \mp \pi i}. \tag{122}$$

At this point, we resolve the sign ambiguity by using each one equally

$$\begin{aligned} \sum_{j=1}^{\infty} e^{-sj\Delta t} e^{\pm \pi ij} &\simeq \frac{1}{2}\left(\frac{1}{s\Delta t + \pi i} + \frac{1}{s\Delta t - \pi i}\right), \\ &= \frac{s\Delta t}{s^2 \Delta t^2 + \pi^2}. \end{aligned} \tag{123}$$

We now rewrite this as

$$\sum_{j=1}^{\infty} e^{-sj\Delta t} e^{\pm \pi ij} \simeq \frac{1}{\Delta t} \frac{s}{s^2 + \frac{\pi^2}{\Delta t^2}}, \tag{124}$$

which we immediately (after consulting a Laplace Transform Table) recognize as

$$L^{-1}\left[\frac{1}{\Delta t} \frac{s}{s^2 + \frac{\pi^2}{\Delta t^2}}\right] = \frac{1}{\Delta t} \cos\left(\frac{\pi t}{\Delta t}\right). \tag{125}$$

This allows us to rewrite Equation (119) as

$$\Delta u \sum_{j=1}^{\infty} e^{\pm \pi i(j+1)}\, \delta\left(t - j\Delta t\right) \simeq -\frac{\Delta u}{\Delta t} \cos\left(\frac{\pi t}{\Delta t}\right), \tag{126}$$

and thus the rate differential equation becomes

$$\frac{d}{dt} u \simeq \frac{\Delta u}{\Delta t} \cos\left(\frac{\pi t}{\Delta t}\right). \tag{127}$$

Before proceeding, it is useful to replace $\Delta t$ with its definition, Equation (111), which now gives us

$$\frac{d}{dt} u \simeq \frac{\Delta u\ u_0}{d} \cos\left(\frac{\pi u_0 t}{d}\right). \tag{128}$$

We now integrate this directly to obtain

$$\begin{aligned} u(t) &\simeq \frac{\Delta u\ u_0}{d}\frac{d}{\pi u_0}\sin\left(\frac{\pi u_0 t}{d}\right)+C \\ &= \frac{\Delta u}{\pi}\sin\left(\frac{\pi u_0 t}{d}\right)+C. \end{aligned} \tag{129}$$

Since the trajectory is closed, we can view this as an orbit and thus anticipate that the constant of integration ($C$) here is zero. Then, since $u(t) = dx(t)/dt$, we may integrate again to obtain the approximate trajectory as

$$\begin{aligned} x(t) &\simeq -\frac{\Delta u}{\pi}\frac{d}{\pi u_0}\cos\left(\frac{\pi u_0 t}{d}\right)+C' \\ &= -\frac{\Delta u d}{\pi^2 u_0}\cos\left(\frac{\pi u_0 t}{d}\right)+C'. \end{aligned} \tag{130}$$

We now apply boundary conditions that the particle start at one side of the box initially ($x(0) = 0$) and be at the opposite side of the box an appropriate time later ($x(\Delta t) = d$). This gives us

$$\begin{aligned} C' - \frac{\Delta u d}{\pi^2 u_0} &= 0, \\ C' + \frac{\Delta u d}{\pi^2 u_0} &= d. \end{aligned} \tag{131}$$

Simple elimination gives us

$$C' = \frac{d}{2}, \tag{132}$$

and

$$\Delta u = \frac{\pi^2 u_0}{2}. \tag{133}$$

This gives us a speed jump factor that is $\pi^2/4$ times what we should have expected based on purely mechanical considerations. The resulting approximate rate trajectory is

$$x(t) \simeq \frac{d}{2}\left[1-\cos\left(\frac{\pi u_0 t}{d}\right)\right], \tag{134}$$

which we compare to the exact trajectory, Equation (113), in Figure 9. The agreement is quite good, largely because of the boundary conditions used.

### B. Bouncing Ball

We are all familiar with the trajectory of a ball that is thrown or dropped, follows a ballistic trajectory impacts the ground, bounces, follows another ballistic trajectory— albeit usually a bit smaller—impacts and bounces again, and so forth. If we neglect air resistance,

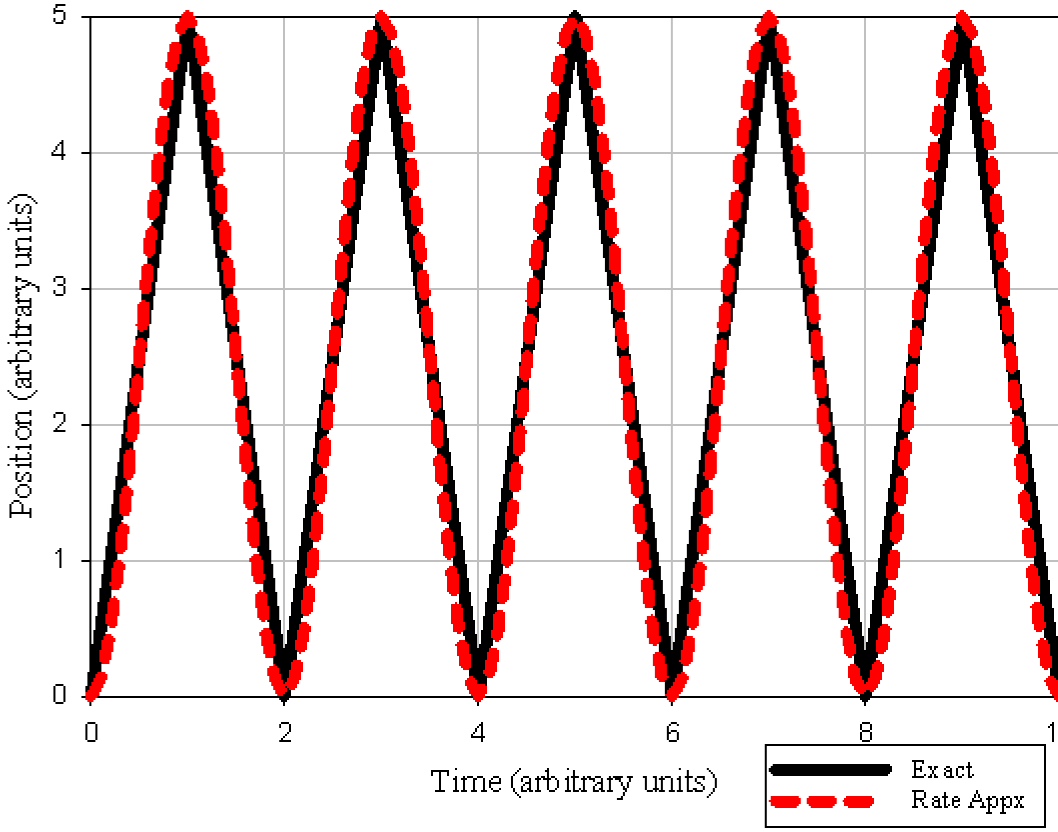


*Figure 9. Comparison of Exact and Rate Pong Trajectories*

which is proportional to either the speed, or the square of the speed, [Marion 1965] and which is small if the speed of the ball is low, then the trajectory of each bounce is given by

$$\begin{aligned} x(t) &= \dot{x}_o t, \\ z(t) &= \dot{z}_0 t - \frac{1}{2} g t^2, \end{aligned} \tag{135}$$

where: $x, z$ are the horizontal, vertical coordinates, $t$ is parametric time, $\dot{x}_o, \dot{z}_0$ are the initial ($t = 0$ measured from the instant when the bounce commences) velocity components, and $g$ is the acceleration due to gravity. The problem, of course, is writing component differential equations (vector differential equation) for the entire trajectory. We approach this problem using our rate theory formalism. As with the Pong example, both jump magnitudes and times are deterministic.

Most discussion of collisions in elementary textbooks is devoted to elastic collisions where energy and momentum are conserved. As we know from observing bouncing ball trajectories, the collisions of the ball with the ground are inelastic since the ball eventually comes to rest. Every time the ball hits the ground, it loses energy. (Unless the ball is made out of Disney

Flubber®, in which case, it gains energy; regardless the formalism is the same.) Four mechanisms may be identified for this:

1. The ball loses a fixed increment of its energy with each collision;
2. The ball loses a fixed fraction of its energy with each collision;
3. The ball's momentum is decreased by a fixed increment with each collision; or
4. The ball's momentum is decreased by a fixed fraction with each collision.

This taxonomy, while simplistic, follows from the form of drag loss. We now consider each of these losses in turn.

### 1. Fixed-Energy Increment

From our development of rate theory methodology, we may write velocity component rate equation primitives as

$$\begin{aligned} \frac{d}{dt}\dot{x} &= \sum_j \Delta\dot{x}_j \delta\left(t-\tau_j\right), \\ \frac{d}{dt}\dot{z} &= \sum_j \Delta\dot{z}_j \delta\left(t-\tau_j\right) - g, \end{aligned} \tag{136}$$

where: $\Delta\dot{x}_j, \Delta\dot{z}_j$ are the horizontal, vertical velocity component jumps and $\tau_j$ are the jump times. The jump times are when the collisions occur and are obviously the same for both components.

We note that while energy and momentum are not conserved during the collisions, they are conserved during the bounces since we are neglecting air drag. Further, mass may be assumed to be conserved. If we adjust the origin of the local coordinate system to each successive collision point, we may place the potential energy zero at that origin so long as the overall collision surface is level. Accordingly, we may write the energy at collision as

$$E = \frac{m}{2}\left(\dot{x}^2 + \dot{z}^2\right), \tag{137}$$

where $m$ is the mass of the ball.

If we now consider this in a circular coordinate system $(r, \theta)$ then immediately around the collision, $\left|\dot{r}\right| \gg \left|r\dot{\theta}\right|$, so we may approximately reduce Equation (137) as

$$E \simeq \frac{m}{2}\dot{r}^2. \tag{138}$$

With this equation, even though energy and thereby momentum are not conserved, we may take momentum as proportionally conserved. That is, the magnitude of momentum is changed in the collision, but the ratios of the momentum components are taken to be conserved. As a result, if we designate the component value just before, after the collision with subscripted minus, plus, we may write proportional conservation relations,

$$\begin{aligned} \frac{\dot{x}_-}{\dot{r}_-} &= \frac{\dot{x}_+}{\dot{r}_+}, \\ \frac{\dot{z}_-}{\dot{r}_-} &= -\frac{\dot{z}_+}{\dot{r}_+}. \end{aligned} \tag{139}$$

The sign change reflects the ball going from moving down to up.

We may invert Equation (138) as

$$\dot{r} \simeq \sqrt{\frac{2E}{m}}, \tag{140}$$

which allows us to rewrite Equations (139) as

$$\begin{aligned} \dot{x}_+ &= \dot{x}_- \frac{\dot{r}_+}{\dot{r}_-} \\ &= \dot{x}_- \sqrt{\frac{E_+}{E_-}}, \\ \dot{z}_+ &= -\dot{z}_- \sqrt{\frac{E_+}{E_-}}. \end{aligned} \tag{141}$$

This gives us a means of calculating the jump magnitudes.

Before proceeding with that, however, it is first useful to number the bounces and designate the energy of each bounce as $E_j, j = 0..$, where the zeroth bounce is the initial one. By Equation (137), the initial energy is just

$$E_0 = \frac{m}{2}\left(\dot{x}_o^{\,2} + \dot{z}_0^{\,2}\right), \tag{142}$$

and the other energies are just

$$E_j = E_0 - j\Delta E, \tag{143}$$

where $\Delta E$ is the constant energy loss per bounce. Quite obviously, the ball cannot have negative energy, so either the energy loss is an integer multiple of the initial energy or the last collision reduces the energy to zero. Either case defines the number of bounces.

Since we are neglecting air drag, as we have already indicated, the ball's trajectory between collisions is conservative. From this we know that the horizontal velocity component is constant over the trajectory element, and the vertical velocity component at the beginning of the element is equal in magnitude and opposite in sign of the component at the end of the element. Let us now designate the velocity components at the beginning of each bounce or trajectory element as $\dot{x}_j, \dot{z}_j$. With this, we may now combine the momentum proportionality equations, Equations (141) with these definitions of energy and velocity components to write equations that relate each bounce to the preceding one

$$\begin{aligned} \dot{x}_j &= \dot{x}_{j-1}\sqrt{\frac{E_j}{E_{j-1}}}, \\ \dot{z}_j &= \dot{z}_{j-1}\sqrt{\frac{E_j}{E_{j-1}}}. \end{aligned} \tag{144}$$

Note that the minus sign in the vertical component relationship disappeared because of the definition of the components.

If we iterate these relationships, we may arrive at a general form

$$\begin{aligned} \dot{x}_j &= \dot{x}_0\sqrt{\frac{E_j}{E_0}}, \\ \dot{z}_j &= \dot{z}_0\sqrt{\frac{E_j}{E_0}}. \end{aligned} \tag{145}$$

We may now use the definition of the element energies, Equation (143) to rewrite these as

$$\begin{aligned} \dot{x}_j &= \dot{x}_0\sqrt{\frac{E_0 - j\Delta E}{E_0}}, \\ \dot{z}_j &= \dot{z}_0\sqrt{\frac{E_0 - j\Delta E}{E_0}}. \end{aligned} \tag{146}$$

Since we are ignoring air drag, we may solve for the collision time increments using the vertical component of Equations (135). This gives us

$$\Delta\tau_j = \frac{2\dot{z}_{j-1}}{g}. \tag{147}$$

We rather obviously reduce this to

$$\begin{aligned} \Delta\tau_j &= \frac{2\dot{z}_0}{g}\sqrt{\frac{E_0 - (j-1)\,\Delta E}{E_0}}, \\ &= \tau_1\sqrt{\frac{E_0 - (j-1)\,\Delta E}{E_0}}, \end{aligned} \tag{148}$$

since $\tau_0 \equiv 0$. This allows us to calculate the jump or collision times as

$$\begin{aligned}\tau_j &= \sum_{k=1}^{j} \Delta\tau_k \\ &= \tau_1 \sum_{k=1}^{j} \sqrt{\frac{E_0 - (k-1)\,\Delta E}{E_0}}.\end{aligned} \tag{149}$$

This series is not obviously summable.

We are now in a position to calculate the velocity component jumps. The horizontal component jump is straightforward,

$$\Delta\dot{x}_j = \dot{x}_j - \dot{x}_{j-1}. \tag{150}$$

This can be slightly elaborated by substituting Equation (146)

$$\Delta\dot{x}_j = \dot{x}_0 \left[ \sqrt{\frac{E_0 - j\Delta E}{E_0}} - \sqrt{\frac{E_0 - (j-1)\,\Delta E}{E_0}} \right]. \tag{151}$$

The vertical component jump is similar,

$$\Delta\dot{z}_j = \dot{z}_j + \dot{z}_{j-1}, \tag{152}$$

except for the sign change resulting from the change in direction of vertical motion across the collision. Elaboration follows directly and we shall not belabor the obvious.

At this point, we close our discussion of this example. Fundamentally, the presence of the index of summation under the radical precludes analytical summation of the series in the jump differential equations, Equations (136). Since analytical forms for the event times and jump magnitudes have been developed, numerical computation of these is straightforward, and thereby numerical simulation of the jump differential equations. Elaboration of this is outside the scope of this presentation.

2. Fixed-Energy Fraction

We now change to a loss representation where the ball's energy is decreased by a fixed fraction $1 - \epsilon$ every time it collides with the ground. This permits us to write the equivalent of Equation (143) as

$$E_j = \epsilon^j E_0. \tag{153}$$

From this, we may proceed to develop the jump magnitudes and jump times.

Our first step is to revise the relationships for the initial velocity components of each bounce, Equations (145). These revisions are

$$\begin{aligned}\dot{x}_j &= \dot{x}_0 \epsilon^{j/2}, \\ \dot{z}_j &= \dot{z}_0 \epsilon^{j/2}.\end{aligned} \tag{154}$$

The jump magnitudes then follow using Equations (150) and (152) as

$$\Delta \overset{\bullet}{x}_j = \overset{\bullet}{x}_0 \epsilon^{j/2} \left(1 - \frac{1}{\epsilon^{\frac{1}{2}}}\right), \tag{155}$$

and

$$\Delta \overset{\bullet}{z}_j = \overset{\bullet}{z}_0 \epsilon^{j/2} \left(1 + \frac{1}{\epsilon^{\frac{1}{2}}}\right). \tag{156}$$

The jump time increments then follow directly from Equations (147) and (154) as

$$\Delta\tau_j = \tau_1 \epsilon^{j/2}. \tag{157}$$

This gives jump times via Equation (149) as

$$\tau_j = \tau_1 \sum_{k=1}^{j} \epsilon^{(k-1)/2}, \tag{158}$$

which we may re-index as

$$\tau_j = \tau_1 \sum_{k=0}^{j-1} \epsilon^{k/2}. \tag{159}$$

We immediately recognize this as a geometric series, permitting us to sum it as

$$\tau_j = \tau_1 \frac{1 - \epsilon^{j/2}}{1 - \epsilon^{1/2}}, \tag{160}$$

giving us an analytical representation of the jump or collision times.

Despite this advance over the previous case, we still do not have a useful rate differential equation. To see this, we take the jump series for the horizontal component in Equation (136) and substitute appropriately,

$$\sum_j \Delta \overset{\bullet}{x}_j \delta\left(t - \tau_j\right) = \overset{\bullet}{x}_0 \left(1 - \frac{1}{\epsilon^{\frac{1}{2}}}\right) \sum_{j=1}^{J^*} \epsilon^{j/2} \delta\left(t - \tau_1 \frac{1 - \epsilon^{j/2}}{1 - \epsilon^{1/2}}\right), \tag{161}$$

where $J^*$ is the number of the last collision when the energy goes to zero. If we now take the Laplace Transform of this equation, we obtain

$$\pounds\left[\sum_j \Delta \overset{\bullet}{x}_j \delta\left(t - \tau_j\right)\right] = \overset{\bullet}{x}_0 \left(1 - \frac{1}{\epsilon^{\frac{1}{2}}}\right) \sum_{j=1}^{J^*} \epsilon^{j/2} \exp\left(-s\tau_1 \frac{1 - \epsilon^{j/2}}{1 - \epsilon^{1/2}}\right). \tag{162}$$

We see immediately that if we attempt to expand the exponential in the manner previously elaborated, we obtain powers of $\frac{1-\epsilon^{j/2}}{1-\epsilon^{1/2}}$ in the denominator of the expansion. How the resulting series may be summed is not obvious. Thus, we abandon further discussion of this case, as we did the previous, for the same reason.

### 3. Fixed-Momentum Increment

We now turn our attention to a case where the loss of energy may be expressed as a loss of momentum by a fixed increment. To pursue this, we return to Equation (139) and make use of this loss of momentum, giving us

$$\dot{x}_{+} = \dot{x}_{-} \frac{\dot{r}_{+}}{\dot{r}_{-}}, \tag{163}$$

and the analogous equation for the vertical component of velocity. Using our previous definitions, we may rewrite this equation as

$$\dot{x}_{j} = \dot{x}_{j-1} \frac{\dot{r}_{j}}{\dot{r}_{j-1}}, \tag{164}$$

and iterating as before produce

$$\dot{x}_{j} = \dot{x}_{0} \frac{\dot{r}_{j}}{\dot{r}_{0}}. \tag{165}$$

If, following the manner of our previous definition, we define

$$\dot{r}_{j} = \dot{r}_{0} - j\Delta\dot{r}, \tag{166}$$

then we may rewrite Equation (165) as

$$\dot{x}_{j} = \dot{x}_{0} \frac{\dot{r}_{0} - j\Delta\dot{r}}{\dot{r}_{0}}. \tag{167}$$

We may now write the horizontal jump magnitudes using this as

$$\begin{aligned} \Delta\dot{x}_{j} &= \dot{x}_{0} \left[ \frac{\dot{r}_{0} - j\Delta\dot{r}}{\dot{r}_{0}} - \frac{\dot{r}_{0} - (j-1)\,\Delta\dot{r}}{\dot{r}_{0}} \right], \\ &= -\dot{x}_{0} \frac{\Delta\dot{r}}{\dot{r}_{0}}. \end{aligned} \tag{168}$$

This is pleasingly simple. The vertical jump magnitudes are not so simple,

$$\begin{aligned} \Delta\dot{z}_{j} &= \dot{z}_{0} \left[ \frac{\dot{r}_{0} - j\Delta\dot{r}}{\dot{r}_{0}} + \frac{\dot{r}_{0} - (j-1)\,\Delta\dot{r}}{\dot{r}_{0}} \right], \\ &= 2\dot{z}_{0} \left[ \frac{\dot{r}_{0} - \left(j - \frac{1}{2}\right)\Delta\dot{r}}{\dot{r}_{0}} \right], \end{aligned} \tag{169}$$

which at least is linear in the index.

We may also calculate the jump time increments from Equation (147) as

$$\begin{aligned}\Delta\tau_j &= \tau_0 \frac{\dot{r}_0 - (j-1)\,\Delta\dot{r}}{\dot{r}_0} \\ &= \tau_0 \left[1 - \frac{(j-1)\,\Delta\dot{r}}{\dot{r}_0}\right].\end{aligned} \tag{170}$$

We may substitute this into Equation (149) as

$$\tau_j = \sum_{k=1}^{j} \tau_0 \left[1 - \frac{(k-1)\,\Delta\dot{r}}{\dot{r}_0}\right], \tag{171}$$

which we may separate and partially re-index as

$$\tau_j = \tau_0 \left[\sum_{k=1}^{j} 1 - \frac{\Delta\dot{r}}{\dot{r}_0} \sum_{k=0}^{j-1} k\right]. \tag{172}$$

Both of these series are summable using basic finite difference methods, giving us

$$\tau_j = \tau_0 \left[j - \frac{\Delta\dot{r}}{\dot{r}_0} \frac{j\,(j-1)}{2}\right]. \tag{173}$$

This form of the jump or collision times is sufficiently complex that it prohibits summation of expansions of the Laplace Transform of the jump series in the rate differential equations. Accordingly, we again cease at efforts short of simple expressions of these differential equations.

4. Fixed-Momentum Fraction

The last case we take up in this example is one where the momentum is reduced by a fixed fraction $1-\mu$ at each collision. As a result, we may form the initial horizontal velocity components as

$$\dot{x}_j = \dot{x}_0 \mu^j, \tag{174}$$

with corresponding equation for the vertical components. Since we have performed this trail of manipulations three times previously, we have omitted several obvious steps.

From this, the horizontal velocity jumps are

$$\Delta\dot{x}_j = \dot{x}_0 \mu^j \left(1 - \frac{1}{\mu}\right), \tag{175}$$

while the vertical velocity jumps are

$$\Delta\dot{z}_j = \dot{z}_0 \mu^j \left(1 + \frac{1}{\mu}\right). \tag{176}$$

The jump time increments are

$$\Delta\tau_j = \tau_0 \mu^{j-1}, \tag{177}$$

and thence the jump times are

$$\tau_j = \sum_{k=1}^{j} \tau_0 \mu^{k-1}. \tag{178}$$

We may re-index this and sum,

$$\begin{aligned} \tau_j &= \tau_0 \sum_{k=0}^{j-1} \mu^k, \\ &= \tau_0 \frac{1-\mu^j}{1-\mu}, \end{aligned} \tag{179}$$

giving us a form essentially the same as we obtained for the constant energy loss increment case.

As a result, we again have a situation where we cannot form simple rate differential equations. While we have defined the problem and elaborated a solution means, reduction to elegant, compact differential equations has eluded us in each case. Nonetheless, we may, in each case, elaborate the primitive rate differential equations and they are readily amenable to numeric simulation.

## X. STOCHASTIC EXAMPLES

We now consider a series of examples where the jump magnitude, or the event times, or both are stochastic.

### A. The Encounter Problem

The encounter problem is one that is common to many disciplines although it may be called by another name. For example, in Physics of War I referred to it as the Target Acquisition problem. I call it the encounter problem here because several disciplines are concerned with meetings between two entities where one or both of the entities are moving, usually randomly. Alternately, one entity may be moving while the other is stationary, and either the motion is random, or the stationary position is random, or both.

Assume that we are concerned with some bounded region of size $\Lambda$, which may be one-, two-, or three-dimensional. Assume that $\Lambda$ is broken up into $N$ subregions of size $\lambda$. Thus, $N\lambda = \Lambda$. Further assume that $\lambda$ is sized so that if an entity is present, it can (but not necessarily must) be detected (observed), but if no entity is present and no detection occurs in a reasonable period of time, progress may continue with reasonable confidence. For elaboration, let there be $m$ $(\ll N)$ entities in $\Lambda$, and assume no $\lambda$ contains more than one entity. Further assume that if there is an entity present, the (discrete) probability it will be detected is $p_d$.

We may treat this as an independent search (sampling) of the $N$ subregions. Thus, the probability that an entity is present in a subregion to be detected on any sample is $m/N$, and the independent probability that an entity is detected on a sample is $p_d\ m/N$. From this, we may write the probability of detecting an entity on the first sample as

$$P(1) = p_d\ \frac{m}{N}, \tag{180}$$

the probability of detecting an entity on the second sample as

$$P(2) = p_d\ \frac{m}{N}\left[1 - p_d\ \frac{m}{N}\right], \tag{181}$$

which is just the probability of no detection by the first sample times the independent probability of detecting an entity on a sample, and the probability of detecting an entity on the third sample as

$$P(3) = p_d\ \frac{m}{N}\left[1 - p_d\ \frac{m}{N}\right]^2. \tag{182}$$

From this, we may generalize to the probability of detecting an entity on the $n^{th}$ sample as

$$P(n) = p_d\ \frac{m}{N}\left[1 - p_d\ \frac{m}{N}\right]^{n-1}. \tag{183}$$

Now, let $\overline{P}(t)$ be the continuous probability that a detection has not occurred by time $t$. Since the process here is repeated (and presumed stochastic), the archetype rate differential equation is given by Equation (106),

$$\frac{d}{dt}\overline{P}(t) = \Delta\overline{P}(t)\ p^{(R)}(t)\ N(t),$$

if the jump function is independent of index. In this case, we take the number of agents to be fixed as one, reducing this equation to

$$\frac{d}{dt}\overline{P}(t) = \Delta\overline{P}(t)\ p^{(R)}(t). \tag{184}$$

If we now examine the discrete formulation above, Equations (180 to 183), we see that we may write the jump function as

$$\Delta\overline{P}(t) = -p_d\ \frac{m}{N}\overline{P}(t), \tag{185}$$

since the discrete term in brackets is just the probability an entity has not been detected.

As a result, we may rewrite the rate differential equation as

$$\frac{d}{dt}\overline{P}(t) = -p_d\ \frac{m}{N}\overline{P}(t)\ p^{(R)}(t). \tag{186}$$

We note immediately that if the repeated process is NED distributed (or is deterministic with equally spaced event times,) then the renewal density function reduces to a constant. Regardless, we may solve this rate differential equation trivially by inspection as

$$\overline{P}(t) = 1 - \exp\left(-\int_0^t p_d \frac{m}{N} p^{(R)}(t')\ dt'\right). \tag{187}$$

Since the quantity we are commonly interested in is the PDF, we obtain that by differentiation as

$$p_{encounter}(t) = p_d \frac{m}{N} p^{(R)}(t) \exp\left(-\int_0^t p_d \frac{m}{N} p^{(R)}(t')\ dt'\right). \tag{188}$$

As before, we note that if the renewal is simple, uniform and NED distributed, the renewal density function is a constant. If the single sample probability of detection, the number of entities, and the number of subregions are constant, then this reduces very neatly to

$$p_{encounter}(t) = p_d \frac{m}{N} p^{(R)}(t) \exp\left(-p_d \frac{m}{N} P^{(R)}(t)\right). \tag{189}$$

If we define the quantity

$$x(t) \equiv \int_0^t p_d \frac{m}{N} p^{(R)}(t')\ dt', \tag{190}$$

then we may change variables and rewrite Equation (188) as

$$p_{encounter}(x) = \exp(-x), \tag{191}$$

from which we may immediately calculate that

$$p^{(R)}_{encounter}(x) = 1. \tag{192}$$

Reversing the change of variable, we obtain

$$p^{(R)}_{encounter}(t) = p_d \frac{m}{N} p^{(R)}(t), \tag{193}$$

which now gives us all the information we need. This is an example of using Rate Theory to convert discrete or mixed stochastic processes into continuous representation.

### B. Drunkard's Walk 1

A dipsomaniac is standing on a sidewalk. Every time increment $\Delta t$, he (she) takes a forward step of length $\Delta x$ with probability $f$, or a backward step (same length) with probability $b$, or stays in place with probability $s$. This process is illustrated in Figure 10.

By inspection, we see that the jump does not depend on index but is sufficiently complicated that some analysis is required. The appropriate rate differential equation is

$$\frac{\partial}{\partial t} u(x,t) = \frac{\Delta u(x,t)}{\Delta t}, \tag{194}$$

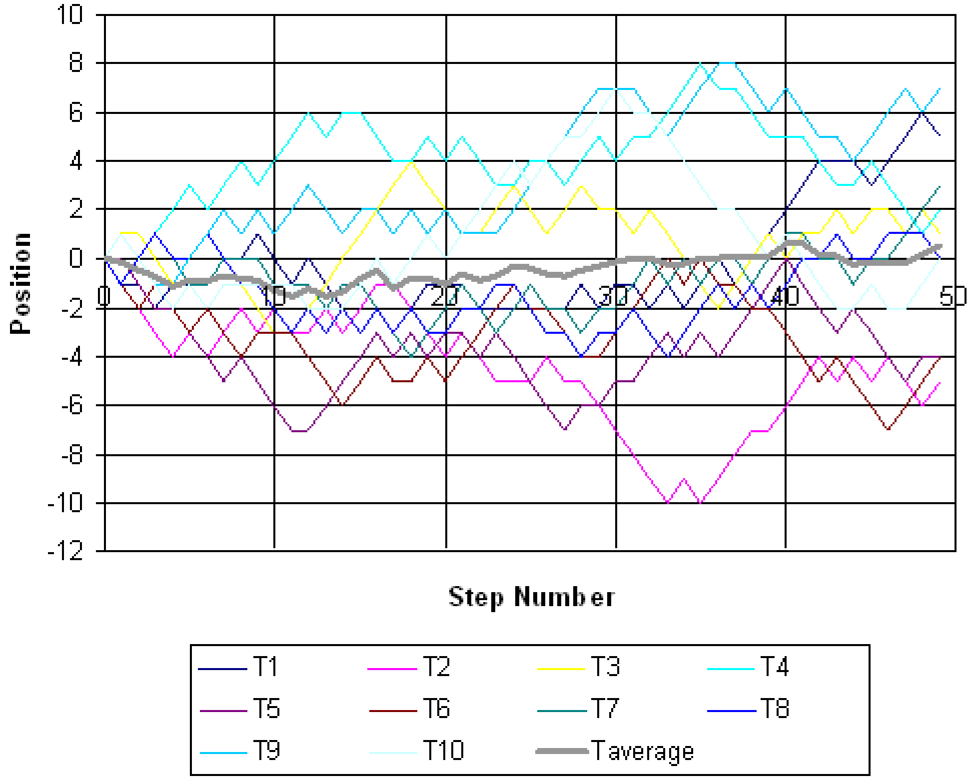


*Figure 10. Illustrative One-Dimensional Drunkard's Walk Trajectories and Mean Trajectory*

where: $u(x,t)$ = probability the dipsomaniac is at position $x$ at time $t$. The challenge is now to form the jump function $\Delta u(x,t)$. Since the jump is presumed to be instantaneous, we may make use of the same continuous representation as we used in the previous example.

Because there are three states – forward, backward, and stay in place – we have to consider four quantities for any position $x$. Prior to the jump, the probability the dipsomaniac is at this position is $u(x,t)$. There is also probability $s$ that this position is maintained. There is also probability $f$ that the dipsomaniac will move from $x$ to $x+\Delta x$, and probability $b$ of moving to $x-\Delta x$. Recognizing that probability flows into the position during the jump, we may write the before jump term as

$$u_{before} = u(x,t)\,, \tag{195}$$

and the after term as

$$u_{after} = f\ u(x-\Delta x,t) + b\ u(x+\Delta x,t) + s\ u(x,t)\,. \tag{196}$$

Notice that the forward and backward probabilities are associated with positions in the opposite directions. This follows from the flow of probability into position $x$ from position $x-\Delta x$ moving forward with probability $f$.

The combination of these two give us the jump function as

$$\begin{aligned}\Delta u(x,t) &= u_{after} - u_{before} \\ &= f\ u(x-\Delta x,t) + b\ u(x+\Delta x,t) + (s-1)\ u(x,t)\,.\end{aligned} \tag{197}$$

We now expand the right side of this equation in a Taylor's Series about $x$. This gives us

$$\begin{aligned}\Delta u(x,t) &= f\ \left[u(x,t) - \frac{\partial u(x,t)}{\partial x}\Delta x + \frac{\partial^2 u(x,t)}{\partial x^2}\frac{\Delta x^2}{2} + HOT\right] \\ &\quad +b\ \left[u(x,t) + \frac{\partial u(x,t)}{\partial x}\Delta x + \frac{\partial^2 u(x,t)}{\partial x^2}\frac{\Delta x^2}{2} + HOT\right] \\ &\quad +(s-1)\ u(x,t)\,.\end{aligned} \tag{198}$$

From conservation of probability, we may simplify this somewhat as

$$\Delta u(x,t) = (b-f)\frac{\partial u(x,t)}{\partial x}\Delta x + (b+f)\frac{\partial^2 u(x,t)}{\partial x^2}\frac{\Delta x^2}{2}, \tag{199}$$

and thence the rate differential equation

$$\frac{\partial}{\partial t}u(x,t) = (b-f)\frac{\Delta x}{\Delta t}\frac{\partial u(x,t)}{\partial x} + \frac{(b+f)}{2}\frac{\Delta x^2}{\Delta t}\frac{\partial^2 u(x,t)}{\partial x^2}. \tag{200}$$

We see immediately that this is the differential equation associated with a diffusion process, the first right-side term being associated with drift and the second with diffusion proper.

### C. Drunkard's Walk 2

The previous example presumed that the steps were taken at even, deterministic intervals. Now we elaborate by assuming that the time when steps occur is stochastic with PDF $p(t)$ and that time to step is independent of step length.

In this case, we should have started with rate differential Equation (106) instead of Equation (103)

$$\frac{\partial}{\partial t}u(x,t) = \Delta u(x,t)\ p^{(R)}(t)\,. \tag{201}$$

The renewal density function can be calculated directly since we know the PDF, and the jump function is the same as developed immediately previous. As a result, we may write the rate differential equation as

$$\begin{aligned}\frac{\partial}{\partial t}u(x,t) &= (b-f)\,\Delta x\ p^{(R)}(t)\,\frac{\partial u(x,t)}{\partial x} \\ &\quad +\frac{(b+f)}{2}\Delta x^2\ p^{(R)}(t)\,\frac{\partial^2 u(x,t)}{\partial x^2},\end{aligned} \tag{202}$$

which we still recognize as a diffusion differential equation. As before, if the time to step PDF is NED, then this rate differential equation reduces back to the form of Equation (200).

## D. Drunkard's Walk 3

We now elaborate when the step length is still independent of time to step but becomes stochastic with PDF $q(x)$. For simplicity, we assume that forward and backward steps are identically distributed. Elaboration to either different distributions or a single distribution for forward and backward steps is straightforward and we shall not treat it here.

Our starting point is Equation (201), and we proceed through the same process of developing the jump function as before. This gives us a rate differential equation

$$\begin{aligned}\frac{\partial}{\partial t}u(x,t) &= (b-f)\,\Delta x\; p^{(R)}(t)\,\frac{\partial u(x,t)}{\partial x}\\ &\quad +\frac{(b+f)}{2}\Delta x^2\; p^{(R)}(t)\,\frac{\partial^2 u(x,t)}{\partial x^2},\end{aligned}$$

which is the same as Equation (202) except that we now interpret $\Delta x$ as a random variable. Accordingly, we need to calculate the expected value of this equation with respect to this random variable. This is straightforward and yields

$$\begin{aligned}\frac{\partial}{\partial t}u(x,t) &= (b-f)\,\langle\Delta x\rangle\; p^{(R)}(t)\,\frac{\partial u(x,t)}{\partial x}\\ &\quad +\frac{(b+f)}{2}\left\langle\Delta x^2\right\rangle\; p^{(R)}(t)\,\frac{\partial^2 u(x,t)}{\partial x^2},\end{aligned} \tag{203}$$

where $\langle\Delta x\rangle, \langle\Delta x^2\rangle$ are the expected values of the step length, step length-squared. The left side is not changed since it is independent of step size. The definition of the standard deviation allows us to rewrite this as

$$\begin{aligned}\frac{\partial}{\partial t}u(x,t) &= (b-f)\,\langle\Delta x\rangle\; p^{(R)}(t)\,\frac{\partial u(x,t)}{\partial x}\\ &\quad +\frac{(b+f)}{2}\left(\sigma^2+\langle\Delta x\rangle^2\right)\; p^{(R)}(t)\,\frac{\partial^2 u(x,t)}{\partial x^2},\end{aligned} \tag{204}$$

which illustrates a difference from what is normally presented in textbooks.

## E. The Logistics Equation

While this section and its differential equation are called the Logistics Equation, I am going to approach it from the standpoint of the spread of disease because I find it easier to understand and develop in that context.

The disease in question is non-lethal – so we don't have to worry about counting those who die – and is spread only through direct contact, which may be either physical contact or near proximity. The quantity $d(t)$ is the number of people at time $t$ who have the disease. We take the total population, which is fixed, to be $N$ and the likelihood that the disease is caught, given contact, is $l$. Obviously, we also assume that no one ever recovers.

All of these restrictions can be relaxed or eliminated, but we are aiming toward a result here and don't want to get too complicated too fast.

The starting point is Equation (106),

$$\frac{d}{dt}d\,(t) = \Delta d\,(t)\;p^{(R)}\,(t)\;N\,(t)\,. \tag{205}$$

We note immediately that the number of agents causing events is just the number of people with the disease, so we may rewrite this immediately as

$$\frac{d}{dt}d\,(t) = \Delta d\,(t)\;p^{(R)}\,(t)\;d\,(t)\,. \tag{206}$$

If an encounter (event) occurs between two diseased people, the jump is zero, since both have the disease; if an encounter occurs between a diseased person and an undiseased person, then the undiseased person may contract the disease with likelihood $l$. Thus, we need only count encounters between diseased and undiseased people and the jump is stochastic with magnitude $l$.

The renewal density function represent the encounters, and since we have a zero jump for encounters between two diseased people, we only have to count encounters between a diseased and an undiseased individual. This is a variant on the Encounter Problem, so we may use Equation (193),

$$p^{(R)}_{encounter}\,(t) = p_d\,\frac{m}{N}\,q^{(R)}\,(t)\,,$$

to build the renewal density function for the Logistics Equation. In this case, the number of subregions is just the total number of people, which is $N \to N$, the number of entities that can be encountered is the number of undiseased people, which is $m \to N - d\,(t)$, the probability of detection is one - $p_d \to 1$,[4] and the sample renewal density rate is just the rate of samples, so $q^{(R)}\,(t) \to q^{(R)}\,(t)$.

As a result, the rate differential equation becomes

$$\frac{d}{dt}d\,(t) = l\frac{N - d\,(t)}{N}\,q^{(R)}\,(t)\;d\,(t)\,. \tag{207}$$

This equation may be rearranged as

$$\frac{d}{dt}d\,(t) = l\,q^{(R)}\,(t)\;d\,(t) - \frac{l\,q^{(R)}\,(t)}{N}\,d^2\,(t)\,, \tag{208}$$

which is a form commonly expressed as the Logistics Equation [Ball 1985].

As a matter of clarification, in this model, the initial condition has to be $d\,(t_0) > 0$. Otherwise, no one ever gets the disease.

[4] Equivalently, we could have counted the encounters not as such but as disease transmission. In that case, the probability of detection would be the likelihood of contracting the disease and the jump function one.

## F. Predator and Prey

The Lotka-Volterra equations,[Ball 1985]

$$\begin{aligned} \frac{dx}{dt} &= Ax - Bxy, \\ \frac{dy}{dt} &= -Cx + Dxy, \end{aligned} \tag{209}$$

are classical differential equations describing the dynamics of a population $x(t)$ of prey and a population $y(t)$ of predators. The first right-side term in the prey equation represents the increase in the prey population, and the second term its decrease. The forms are reversed in the predator equation.

We assume that these prey and predator constitute a closed system and that there is sufficient food for the prey that they never lack.

In a closed system where members do not migrate, an increase in the number of members is the result of reproduction. Reproduction may take two forms (that we know of): sexual and asexual. In asexual reproduction, one parent divides into two or more offspring. In sexual reproduction, two parents must mate to produce one or more offspring, $n$.

In terms of rates, asexual reproduction produces $n$ offspring, where $n$ may be the expected value of some random variable, so we have a rate of increase of asexually reproducing population as

$$r_{asexual} \sim \frac{n-1}{\Delta t} N(t), \tag{210}$$

where: $\Delta t$ = the time between reproductions (assuming these times are deterministic or stochastic with NED distribution, otherwise, $= 1 \big/ s^{(R)}(t)$, $s^{(R)}(t)$ = renewal density function of reproductions), and $N(t)$ = (reproducible) population at time $t$. If, on the other hand, the reproduction is sexual, then the rate of that reproduction should be

$$r_{sexual} \sim \frac{n}{2\Delta t} N(t)^2, \tag{211}$$

on the assumption that the population is evenly divided between male and female members. (Whence the factor of 2, since only half of possible encounters will be between opposite sex individuals.) The form of this rate follows directly from the encounter problem.

There are two major causes of death for prey and one for predators. Both may die of old age, which has a rate of

$$r_{death} \sim -\frac{1}{\Delta t} N(t), \tag{212}$$

where: $\Delta t$ = the average life span (again either deterministic or NED). This assumes we have not been keeping up with ages and death by old age occurs uniformly among the population (which is a bit of a misnomer.) Death by being eaten, which applies uniquely to prey, has the rate

$$r_{eaten} \sim -\frac{p}{\Delta t} N(t) M(t), \tag{213}$$

where: $\Delta t =$ the encounter time for predator and prey, $p =$ the probability that a predator will catch and eat a prey (actually kill a prey) in an encounter, and $N(t), M(t) =$ the populations of prey and predator.

If we now assume that both predator and prey reproduce asexually, then the predator-prey rate differential equations should be

$$\begin{aligned} \frac{dx}{dt} &= \frac{n-1}{\Delta t_{repro}} x - \frac{1}{\Delta t_{old\ age}} x - \frac{p}{\Delta t_{eaten}} xy, \\ \frac{dy}{dt} &= \frac{n'-1}{\Delta t'_{repro}} y - \frac{1}{\Delta t'_{old\ age}} y, \end{aligned} \tag{214}$$

since eating prey does not cause new predators directly to be produced. Note that the predator quantities are primed here. If both predators and prey reproduce sexually, then the rate differential equations may be

$$\begin{aligned} \frac{dx}{dt} &= \frac{n}{2\Delta t_{repro}} x^2 - \frac{1}{\Delta t_{old\ age}} x - \frac{p}{\Delta t_{eaten}} xy, \\ \frac{dy}{dt} &= \frac{n'}{2\Delta t'_{repro}} y^2 - \frac{1}{\Delta t'_{old\ age}} y. \end{aligned} \tag{215}$$

Notice that neither set of these rate differential equations look like the Lotka-Volterra equations.

We might want to account for a predator starving to death if it doesn't get to eat a prey in a certain amount of time. If we say that a predator will starve if it goes $k$ encounters with a prey without eating. If the time between encounters is $\Delta t_{encounter}$ and the probability of eating a prey on an encounter is $\sim p_{eat} x / \xi$, where $\xi =$ the ratio of total hunting area to encounter area, then the rate of starvation is approximately

$$r_{starve} \sim -\frac{1}{k\Delta t_{encounter}} \left[1 - \frac{p_{eat} x}{\xi}\right]^k y, \tag{216}$$

and the sexual rate differential equations become

$$\begin{aligned} \frac{dx}{dt} &= \frac{n}{2\Delta t_{repro}} x^2 - \frac{1}{\Delta t_{old\ age}} x - \frac{p}{\Delta t_{eaten}} xy, \\ \frac{dy}{dt} &= \frac{n'}{2\Delta t'_{repro}} y^2 - \frac{1}{\Delta t'_{old\ age}} y - \frac{1}{k\Delta t_{encounter}} \left[1 - \frac{p_{eat} x}{\xi}\right]^k y. \end{aligned} \tag{217}$$

If $p_{eat} x / \xi \ll 1$, then we have the forms

$$\begin{aligned} \frac{dx}{dt} &= \frac{n}{2\Delta t_{repro}} x^2 - \frac{1}{\Delta t_{old\ age}} x - \frac{p}{\Delta t_{eaten}} xy, \\ \frac{dy}{dt} &= \frac{n'}{2\Delta t'_{repro}} y^2 - \frac{1}{\Delta t'_{old\ age}} y - \frac{\left[1 - \frac{k p_{eat} x}{\xi}\right]}{k\Delta t_{encounter}} y, \end{aligned} \tag{218}$$

which gets us closer to the form of the Lotka-Volterra equations, especially for the asexual reproduction forms.

This is a rather obvious example where rate theory produces something strikingly different from what is commonly used on a more *ad hoc* basis.

### G. Lanchester Equations

The Lanchester Equations [Lanchester 1916] describe losses to a force in combat due to the exchange of fire. An elaboration of these equations is commonly used to describe forces comprised of more than one type of combat element, but we shall restrict our discussion here to the homogeneous problem. Considerable discussion of the validity of these differential equations rages, in many cases because of the lack of understanding of what they represent. We derive these equations to their commonly used (and restricted) forms.

The Lanchester Equations are normally paired and symmetric, although this is not necessary and is more convenience and tradition than necessity. Additionally, while they admit (at least for one form) changes in "force strength" due to reinforcement, they are commonly started impulsively while real combat is seldom so ordered [Fowler 2004].

We consider two cases where each individual weapons' fire has a lethality area/volume that is less than that of a single combat element or more than that of one. Thus, we distinguish between situations of point lethality where at most one combat element may be attrited and area (volume) lethality where more than one combat element may be attrited.

We label the two sides as enemy and friendly, and designate the number of elements at time $t$ as $E(t)$ for the enemy force, and $F(t)$ for the friendly force. We also limit our discussion to only one side's suffered attrition.

#### 1. Point Lethality

Point lethality is commonly associated with direct, that is, line-of-sight, fire. It commonly consists of two processes: finding an enemy element to shoot at and the actual shooting itself. In most cases, the first is a variant of the encounter problem. We can approximate the target-finding process with a PDF ala Equation (188)

$$\begin{aligned} p_{encounter}(t) &= p_d \frac{m}{N} p^{(R)}(t) \exp\left(-\int_0^t p_d \frac{m}{N} p^{(R)}(t')\ dt'\right) \\ &\rightarrow p_{find}(t) \simeq \alpha \frac{E(t)}{A(t)} \exp\left(-\alpha \frac{E(t)}{A(t)} t\right), \end{aligned} \tag{219}$$

where: $A(t)$ = the area (or volume, depending on the search and sensor forms,) occupied by the enemy force, the search process is assumed to be uniform, simple, and NED, and the ratio $E(t)/A(t)$ is assumed to be constant or slowly varying. Since we will be leading up to some limiting behaviors, the latter is not critical to the development of Lanchester's Equations but is in the general context of attrition modeling. Also, because of these restrictions and assumptions, all other factors may be lumped in the coefficient $\alpha$. Note that we may develop $A(t)$ from the definition of $\Lambda$ and $\lambda$ in our discussion of the encounter problem above.

The shooting process is assumed to be NED with rate $\beta$. As a result, we may write the point-lethality process renewal density function as

$$p_{Point}^{(R)}(t) = \frac{\alpha \frac{E(t)}{A(t)}\beta}{\alpha \frac{E(t)}{A(t)} + \beta}\left[1 - e^{-\left(\alpha \frac{E(t)}{A(t)}+\beta\right)t}\right], \tag{220}$$

from Equation (109). (We are considering here the case of friendly elements shooting at enemy elements.)

Since this is point lethality, the jump is uniquely $-1$, since only one element is attrited and the attrition reduces the force strength. Thus, from Equation (106) we may write the rate differential equation of the enemy force strength as

$$\frac{d}{dt}E(t) = -\frac{\alpha \frac{E(t)}{A(t)}\beta}{\alpha \frac{E(t)}{A(t)} + \beta}\left[1 - e^{-\left(\alpha \frac{E(t)}{A(t)}+\beta\right)t}\right]F(t). \tag{221}$$

We now wish away the time dependent term in the brackets. Two related but different arguments permit this. The first, which is similar to that used in Bonder-Farrell Theory [Fowler 2004, and references therein] is that the rate is defined ala Blackwell's Theorem in the infinite time limit. The second is that we are interested in the steady state or equilibrium behavior. Regardless, Equation (221) simplifies to

$$\frac{d}{dt}E(t) = -\frac{\alpha \frac{E(t)}{A(t)}\beta}{\alpha \frac{E(t)}{A(t)} + \beta}F(t). \tag{222}$$

We now assume that the time to find a target is much less than the time to shoot and attrit a target. This is just an investigation of limiting behavior. As a result,

$$\alpha \frac{E(t)}{A(t)} \gg \beta, \tag{223}$$

and Equation (222) may be reduced to

$$\frac{d}{dt}E(t) = -\beta F(t), \tag{224}$$

which is a linear differential equation, but in Lanchester Theory terms it is called a quadratic Lanchester equation (because of the form of it and its symmetric complement's normal form solution). This demonstrates that point lethality reduces to a quadratic Lanchester differential equation when target density is high, target shooting is slow, or target finding is fast.

When the opposite is true, and the time to find a target is much greater than the time to shoot and attrit a target, Equation (222) reduces to

$$\frac{d}{dt}E(t) = -\alpha \frac{E(t)}{A(t)}F(t). \tag{225}$$

Now, if the density of enemy elements stays fixed over time, this further reduces to

$$\frac{d}{dt}E(t) = -\alpha \frac{E(0)}{A(0)}F(t), \tag{226}$$

which is again a quadratic Lanchester differential equation. If the target density varies, as would be the case if the area occupied stays fixed, then Equation (222) reduces to

$$\frac{d}{dt}E(t) = -\alpha \frac{E(t)}{A(0)}F(t), \tag{227}$$

which is a non-linear rate differential equation, but in Lanchester terms it is a linear Lanchester differential equation. This demonstrates the other extreme behavior.

### 2. Area Lethality

Area lethality fire is normally directed since the weapons do not have line-of-sight to their targets. In such circumstances and assuming that the firing is simple, uniform, and NED with rate $\gamma$, we may again use Equation (106)

$$\frac{d}{dt}E(t) = \Delta E \gamma \; F(t). \tag{228}$$

The jump is the number of enemy elements killed per friendly shot. This should be (approximately)

$$\Delta E \simeq -p_{k|s} \frac{E(t)}{A(t)} a_{Lethal}, \tag{229}$$

where: $p_{k|s}$ = the probability of kill in the lethal area given a shot, $a_{Lethal}$ = lethal area per shot, and $E(t)/A(t)$ = the density of enemy elements. Substitution of this yields

$$\frac{d}{dt}E(t) = -p_{k|s} \frac{E(t)}{A(t)} a_{Lethal}\gamma \; F(t). \tag{230}$$

If we now take the limit that the density stays constant, then this reduces to

$$\frac{d}{dt}E(t) = -p_{k|s} \frac{E(0)}{A(0)} a_{Lethal}\gamma \; F(t), \tag{231}$$

which is a quadratic Lanchester differential equation. If the occupied area stays fixed, then the limit is

$$\frac{d}{dt}E(t) = -p_{k|s} \frac{E(t)}{A(0)} a_{Lethal}\gamma \; F(t), \tag{232}$$

which is a linear Lanchester differential equation.

Regardless of whether we are considering point or area lethality in the long time or equilibrium limit, we may obtain either a linear or a quadratic Lanchester differential equation, depending on limiting assumptions.

## XI. REFERENCES


| | |
|---|---|
| Abate 1997. | Abate, Frank R., chief editor, **The Oxford Desk Dictionary**, Berkley Books, New York, 1997. |
| Ball 1985. | Ball, M. A., **Mathematics in the Social Sciences and Life Sciences: Theories, Models, and Methods**, John Wiley and Sons, New York, 1985. |
| Cox Smith 1954. | Cox, D. R., and Smith, Walter L. "On the Superposition of Renewal Processes," *Biometrika*, Vol 41, 1954, pp. 91-99. |
| David 1970. | David, H. A., **Order Statistics**, John Wiley and Sons, New York, 1970. |
| Doetsch 1961. | Doetsch, Gustav, Guide to the Applications of the Laplace and $\zeta$-Transforms, Van Nostrand Reinhold Co., London, 1961. |
| Erdelyi 1962. | Erdelyi, Arthur, **Operational Calculus and Generalized Functions**, Holt, Rinehart and Winston, New York, 1962. |
| Fowler 2001. | Fowler, Bruce W., "Attrition Rate Theory, Phase Aggregation, and Formal Aggregation," in Fowler, Bruce W., ed., **Proceedings of the Millennium Attrition Symposium**, 4-7 August 2001, Houghton, MI, in preparation. |
| Fowler 2004. | Fowler, Bruce W., **The Physics of War**, in press. |
| Gradshteyn Ryzhik 1980. | Gradshteyn, I. S., and Ryzhik, I. M., **Tables of Integrals, Series, and Products**, Academic Press, New York, 1980. |
| Goldstein 1950. | Goldstein, Herbert, **Classical Mechanics**, Addison-Wesley Publishing Company, Ltd., Cambridge, 1950. |
| Lanchester 1916. | Lanchester, Frederick W., **Aircraft in Warfare: The Dawn of the Fourth Arm**, Constable and Company, LTD., London, 1916. reprinted by the Lanchester Press, Inc., Sunnyvale, 1995. |
| Macrobert 1962. | MacRobert, Thomas Murray, **Functions of a Complex Variable**, MacMillan, London; St. Martin's Press, New York, 1962. |
| Marion 1965. | Marion, Jerry B., **Classical Dynamics of Particles and Systems**, Academic Press, New York, 1965, pp. 64-73. |
| Medhi 1994. | Medhi, J., **Stochastic Processes**, John Wiley and Sons, Inc., New York, 1994. |
| Mohling 1982. | Mohling, Fritz, **Statistical Mechanics**, John Wiley and Sons, Inc., New York, 1982. |

Ross 1970. Ross, Sheldon M., **Applied Probability Models with Optimization Applications**, Dover, New York, 1992.

Stakgold 1998. Stakgold, Ivar, **Green's Functions And Boundary Value Problems**, John Wiley and Sons, New York, 1998.

Taylor Karlin 1998. Taylor, Howard M., and Karlin, Samuel, **An Introduction to Stochastic Modeling**, Academic Press, New York, 1998.

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